p. 745: 11, 20-22, 31-33, 64-68, 73-77, 79-83 odd, 90-92, 119-121

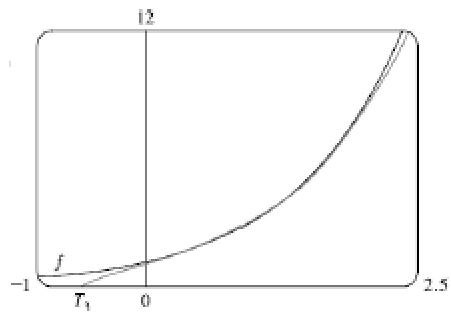
11. Because $f^{(n)}(4)=\dfrac{(-1)^n n!}{3^n(n+1)}$, the Taylor series for f centered at 4 is $\sum_{n=0}^{\infty}\dfrac{f^{(n)}(4)}{n!}(x-4)^n$

$=\sum_{n=0}^{\infty}\dfrac{(-1)^n n!}{3^n(n+1)n!}(x-4)^n=\sum_{n=0}^{\infty}\dfrac{(-1)^n}{3^n(n+1)}(x-4)^n$. Apply the Ratio Test to find the radius of convergence R: $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\dfrac{(-1)^{n+1}(x-4)^{n+1}}{3^{n+1}(n+2)}\cdot\dfrac{3^n(n+1)}{(-1)^n(x-4)^n}\right|=\lim_{n\to\infty}\left|\dfrac{(-1)(x-4)(n+1)}{3(n+2)}\right|$

$=\dfrac{1}{3}|x-4|\cdot\lim_{n\to\infty}\dfrac{n+1}{n+2}=\dfrac{1}{3}|x-4|$. For convergence, $\frac{1}{3}|x-4|<1 \Leftrightarrow |x-4|<3$, so $R=3$.

20.

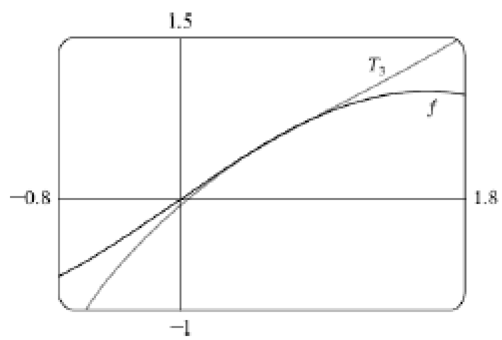
n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	e^x	e
1	e^x	e
2	e^x	e
3	e^x	e



$$T_3(x)=\sum_{n=0}^{3}\frac{f^{(n)}(1)}{n!}(x-1)^n=\frac{e}{0!}(x-1)^0+\frac{e}{1!}(x-1)^1+\frac{e}{2!}e(x-1)^2+\frac{e}{3!}e(x-1)^3$$
$$=e+e(x-1)+\tfrac{1}{2}e(x-1)^2+\tfrac{1}{6}e(x-1)^3$$

21.

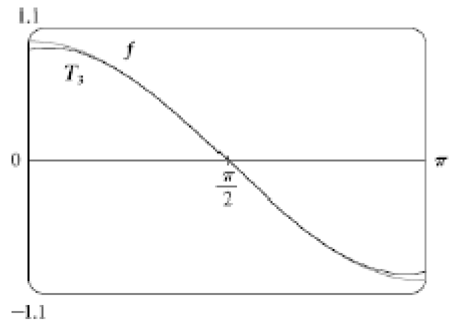
n	$f^{(n)}(x)$	$f^{(n)}(\pi/6)$
0	$\sin x$	$\frac{1}{2}$
1	$\cos x$	$\frac{\sqrt{3}}{2}$
2	$-\sin x$	$-\frac{1}{2}$
3	$-\cos x$	$-\frac{\sqrt{3}}{2}$



$$T_3(x)=\sum_{n=0}^{3}\frac{f^{(n)}(\pi/6)}{n!}\left(x-\tfrac{\pi}{6}\right)^n=\frac{1/2}{0!}\left(x-\tfrac{\pi}{6}\right)^0+\frac{\sqrt{3}/2}{1!}\left(x-\tfrac{\pi}{6}\right)^1-\frac{1/2}{2!}\left(x-\tfrac{\pi}{6}\right)^2-\frac{\sqrt{3}/2}{3!}\left(x-\tfrac{\pi}{6}\right)^3$$
$$=\frac{1}{2}+\frac{\sqrt{3}}{2}\left(x-\tfrac{\pi}{6}\right)-\frac{1}{4}\left(x-\tfrac{\pi}{6}\right)^2-\frac{\sqrt{3}}{12}\left(x-\tfrac{\pi}{6}\right)^3$$

22.

n	$f^{(n)}(x)$	$f^{(n)}(\pi/2)$
0	$\cos x$	0
1	$-\sin x$	-1
2	$-\cos x$	0
3	$\sin x$	1



$$T_3(x)=\sum_{n=0}^{3}\frac{f^{(n)}(\pi/2)}{n!}\left(x-\tfrac{\pi}{2}\right)^n=-\left(x-\tfrac{\pi}{2}\right)+\tfrac{1}{6}\left(x-\tfrac{\pi}{2}\right)^3$$

31.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\ln(1+x)$	0
1	$(1+x)^{-1}$	1
2	$-(1+x)^{-2}$	-1
3	$2(1+x)^{-3}$	2
4	$-6(1+x)^{-4}$	-6
5	$24(1+x)^{-5}$	24
⋮	⋮	⋮

$$\ln(1+x)=\sum_{n=0}^{\infty}\frac{f^{(n)}0)}{n!}x^n=0+x-\tfrac{1}{2}x^2+\tfrac{2}{6}x^3-\tfrac{6}{24}x^4+\tfrac{24}{120}x^5+\cdots$$

$$=x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+\tfrac{1}{5}x^5+\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)}\cdot\frac{n}{x^n}\right|=\lim_{n\to\infty}\frac{|x|}{1+1/n}=|x|<1\Leftrightarrow R=1.$$

32.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cos x$	1
1	$-\sin x$	0
2	$-\cos x$	-1
3	$\sin x$	0
4	$\cos x$	1
⋮	⋮	⋮

$$\cos x=\sum_{n=0}^{\infty}\frac{f^{(n)}0)}{n!}x^n=1-\tfrac{1}{2!}x^2+\tfrac{1}{4!}x^4-\cdots=\sum_{n=0}^{\infty}(-1)^{n-1}\frac{x^{2n}}{(2n)!}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{x^{2n}}\right|=\lim_{n\to\infty}\frac{x^2}{(2n+2)(2n+1)}=0<1$$ for all x, so $R=\infty$.

33.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	e^{-2x}	1
1	$-2e^{-2x}$	-2
2	$4e^{-2x}$	4
3	$-8e^{-2x}$	-8
4	$16e^{-2x}$	16
⋮	⋮	⋮

$$e^{-2x}=\sum_{n=0}^{\infty}\frac{f^{(n)}0)}{n!}x^n=\sum_{n=0}^{\infty}\frac{(-2)^n}{n!}x^n.$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(-2)^{n+1}x^{n+1}}{(n+1)!}\cdot\frac{n!}{(-2)^n x^n}\right|=\lim_{n\to\infty}\frac{2|x|}{n+1}=0<1$$ for all x, so $R=\infty$.

64. $\arctan x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$, so $f(x)=\arctan(x^2)=\sum_{n=0}^{\infty}(-1)^n\frac{(x^2)^{2n+1}}{2n+1}x^{4n+2}=\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}x^{4n+2}$, and $R=1$.

65. $\sin(x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$, so $f(x)=\sin\left(\frac{\pi}{4}x\right)=\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{\pi}{4}x)^{2n+1}}{4^{2n+1}(2n+1)!}=\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{4^{2n+1}(2n+1)!}x^{2n+1}$, and $R=\infty$.

66. $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \Rightarrow \cos 2x = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} x^{2n}}{(2n)!}$, so

$f(x) = x\cos 2x = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}}{(2n)!} x^{2n+1}$, with $R = \infty$.

67. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, so $f(x) = e^{3x} - e^{2x} = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} - \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} - \sum_{n=0}^{\infty} \frac{2^n x^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n - 2^n}{n!} x^n$, $R = \infty$.

68. $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \Rightarrow \cos\left(\frac{1}{2}x^2\right) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{2}x^2\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{2^{2n}(2n)!}$, so

$f(x) = x\cos\left(\frac{1}{2}x^2\right) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{2n}(2n)!} x^{4n+1}$, $R = \infty$.

73. $\frac{x - \sin x}{x^3} = \frac{1}{x^3}\left[x - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right] = \frac{1}{x^3}\left[x - x - \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right] = \frac{1}{x^3}\left[-\sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!}\right]$

$= \frac{1}{x^3}\left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n+3)!}\right] = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+3)!}$, and this series also gives the required value at $x = 0$ (namely 1/6). $R = \infty$.

74. $x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \cdots = x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) = x\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x\sin x$, option (**C**).

75. We know $e^{x^2} = 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$ and so $e^{x^2} - 1 = \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$. Therefore, choice (**B**) is correct: $\frac{e^{x^2} - 1}{x} = \frac{\frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots}{x} = x + \frac{x^3}{2!} + \frac{x^5}{3!} + \cdots,$.

76. We know $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ and $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$. Therefore,

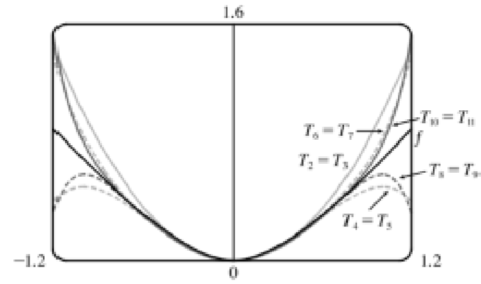
$e^x \cos x = (1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots)(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots) = (1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^2 - \frac{1}{2}x^3 + \frac{1}{3}x^3 + \cdots)$

$= 1 + x - \frac{1}{3}x^3 - \frac{1}{6}x^4 + \cdots$ which is option (**D**).

77. Using Table 9.5 we find $\cos(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!}$ so

$x^2\cos(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^2 x^{4n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n)!}$, which is the series in (**D**).

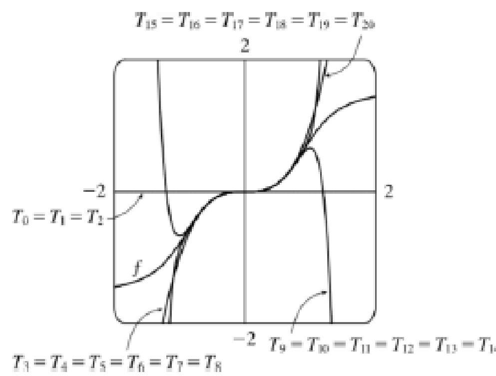
79. $\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$



$$\ln(1+x^2)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(x^2)^n}{n}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n}}{n}$$

$=x^2-\frac{1}{2}x^4+\frac{1}{3}x^6-\frac{1}{4}x^8+\cdots$. The series for $\ln(1+x)$ has $R=1$ and $|x^2|<1 \Leftrightarrow |x|<1$, so the series for $f(x)$ also has $R=1$. From the graphs of f and the first few Taylor polynomials, we see that $T_n(x)$ provides a closer fit to $f(x)$ near 0 as n increases.

81. $\tan^{-1}x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$, so $f(x)=\tan^{-1}(x^3)$



$$=\sum_{n=0}^{\infty}(-1)^n\frac{(x^3)^{2n+1}}{2n+1}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{6n+3}}{2n+1}$$

$$=x^3-\tfrac{1}{3}x^9+\tfrac{1}{5}x^{15}-\tfrac{1}{7}x^{21}+\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{(n-1)!}$$

The series for $\tan^{-1}x$ has $R=1$ and $|x^3|<1 \Leftrightarrow |x|<1$, so the series for $f(x)$ also has $R=1$. From the graphs of f and the first few Taylor polynomials, we see that $T_n(x)$ provides a closer fit to $f(x)$ near 0 as n increases.

83. $1/\sqrt[10]{e}=e^{-1/10}$ and $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$ so,

$e^{-1/10}=1-\frac{1}{10}+\frac{(1/10)^2}{2!}-\frac{(1/10)^3}{3!}+\frac{(1/10)^4}{4!}-\frac{(1/10)^5}{5!}+\cdots$. Now

$1-\frac{1}{10}+\frac{(1/10)^2}{2!}-\frac{(1/10)^3}{3!}+\frac{(1/10)^4}{4!}\approx 0.90484$ and subtracting $\frac{(1/10)^5}{5!}\approx 8.3\times 10^{-8}$ does not affect the fifth decimal place, so $e^{-1/10}\approx 0.90484$ by the Alternating Series Estimation Theorem.

90. $\sin x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\Rightarrow \sin(x^2)=0\sum_{n=1}^{\infty}(-1)^n\frac{(x^2)^{2n+1}}{(2n+1)!}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{(2n+1)!}\Rightarrow$

$x^2\sin(x^2)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+4}}{(2n+1)!}\Rightarrow \int x^2\sin(x^2)\,dx=C+\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+5}}{(2n+1)!(4n+5)}$, with $R=\infty$.

91. $\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\Rightarrow \cos x-1=\sum_{n=1}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\Rightarrow \frac{\cos x-1}{x}=\sum_{n=1}^{\infty}(-1)^n\frac{x^{2n-1}}{(2n)!}\Rightarrow$

$\int\frac{\cos x-1}{x}\,dx=C+\sum_{n=1}^{\infty}(-1)^n\frac{x^{2n}}{2n\cdot(2n)!}$, with $R=\infty$.

92. $\arctan x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}\Rightarrow \arctan(x^2)=\sum_{n=0}^{\infty}(-1)^n\frac{(x^2)^{2n+1}}{2n+1}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{2n+1}\Rightarrow$

$\int\arctan(x^2)\,dx=C+\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+3}}{(2n+1)(4n+3)}$, with $R=1$.

119. $a_3 = \frac{f^{(3)}(5)}{3!} = \frac{(-1)^3 \cdot 3!}{3! \cdot 2^3(3+2)} = \frac{-1}{8 \cdot 5} = -\frac{1}{40}$, choice (**C**).

120. $f'(x) = 2e^{2x} + 2x\sin(x^2)$, $f''(x) = 4e^{2x} + 2\sin(x^2) + 4x^2\cos(x^2)$,
$f'''(x) = 8e^{2x} + 12x\cos(x^2) - 8x^3\sin(x^2)$, and
$f^{(4)}(x) = 16e^{2x} + 12\cos(x^2) - 48x^2\sin(x^2) - 16x^4\cos(x^2)$. $f^{(4)}(0) = 16 + 12 = 28$, so the coefficient of x^4 in the Taylor series for f centered at 0 is $\frac{f^{(4)}(0)}{24} = \frac{28}{24} = \frac{7}{6}$, choice (**A**).

121. $f'(x) = \sum_{n=1}^{\infty} (-1)^n \frac{(3x)^{2n+1}}{(2n+1)!} = \sin(3x) \Rightarrow f(x) = \int \sin(3x)\,dx = -\frac{\cos(3x)}{3} + C$. Using $f(0) = 2$ we find $C = 2$, so $f(x) = 2 - \frac{\cos(3x)}{3}$. Then, $f\left(\frac{\pi}{3}\right) = 2 - \frac{\cos(\pi)}{3} = 2 - \left(\frac{1}{3}\right) = \frac{7}{3}$, choice (**B**).