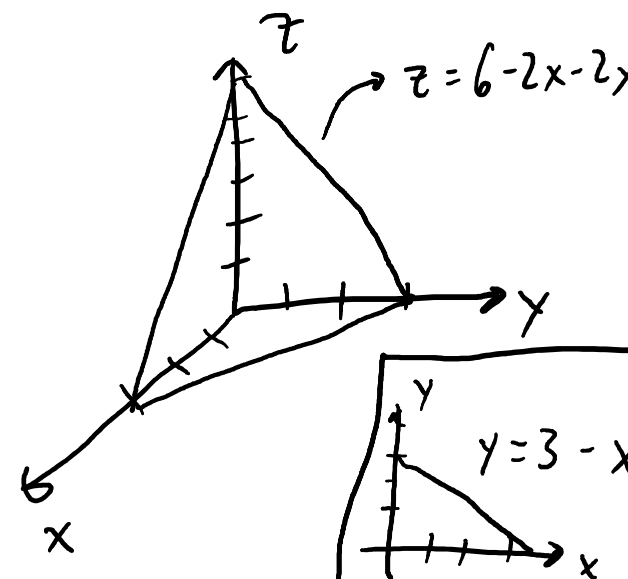
# Some Theorems

Thm. Divergence Theorem

Let $E$ be a solid region and let $S$ be the boundary surface of $E$. Let **F** be a vector field whose component functions have continuous partial derivatives on an open region containing $E$. Then

$$\iint_S \overrightarrow{F} \cdot d\overrightarrow{S} = \iiint_E \operatorname{div} \overrightarrow{F}\, dV$$

Ex. Evaluate $\iint_S \vec{F}\cdot d\vec{S}$ where $\mathbf{F} = x\mathbf{i} + 2y\mathbf{j} + z\mathbf{k}$ and $S$ is the surface of the solid bounded by $2x + 2y + z = 6$ and the coordinate planes.




$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\,dV$$

$$= \int_0^3 \int_0^{3-x} \int_0^{6-2x-2x} 4 \, dz\,dy\,dx$$

$$\operatorname{div}\vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle x, 2y, z \rangle$$

$$= 1 + 2 + 1 = 4$$

Ex. Find the flux of the vector field $\mathbf{F} = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$ over the unit sphere.

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iiint_E \operatorname{div} \vec{F}\, dV = \iiint_E (0+1+0)\, dV = \iiint_E 1\, dV$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^{1} \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$\downarrow$$

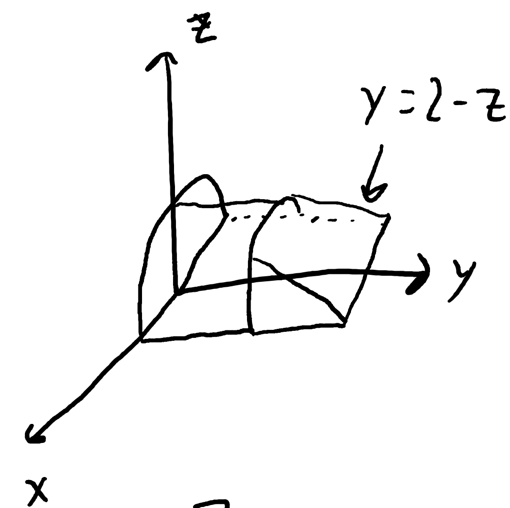
$$V_{sphere}$$

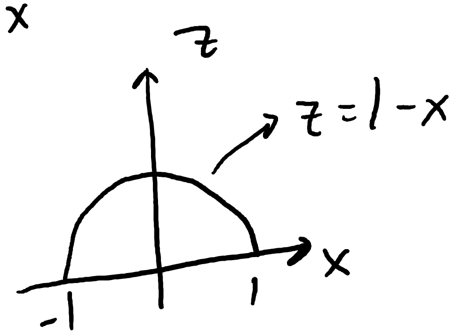
$$\frac{4\pi}{3}$$

<u>Ex.</u> Evaluate $\iint_S \vec{F}\cdot d\vec{S}$ where $\mathbf{F} = xy\mathbf{i} + (y^2 + e^{xz})\mathbf{j} + \ln(xy)\mathbf{k}$ and $S$ is the surface of the region bounded by $z = 1 - x^2$, $z = 0$, $y = 0$, and $y + z = 2$

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\,dV$$

$$= \int_{-1}^{1}\int_{0}^{1-x^2}\int_{0}^{2-z} (y + 2y + 0)\,dy\,dz\,dx$$

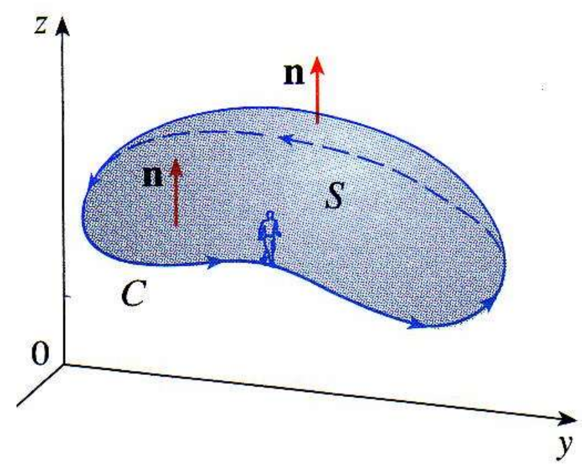


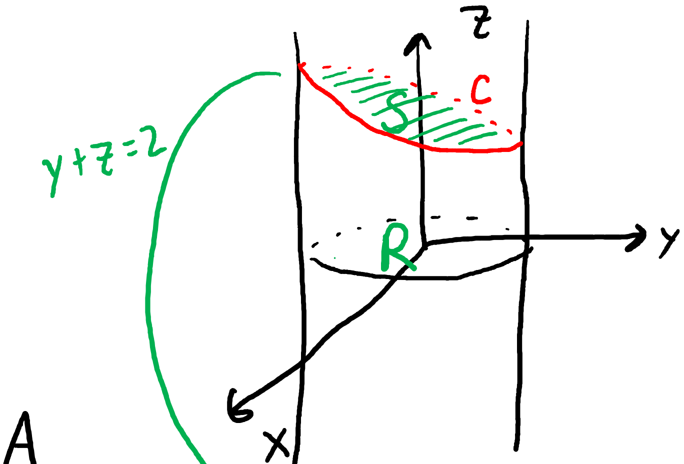
Thm. Stokes' Theorem

Let $S$ be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth curve $C$. Let **F** be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains $S$. Then

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$$



Ex. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\mathbf{F} = -y^2\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$ and $C$ is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$.




$S: g(x,y) = 2-y$

$R =$ unit circle

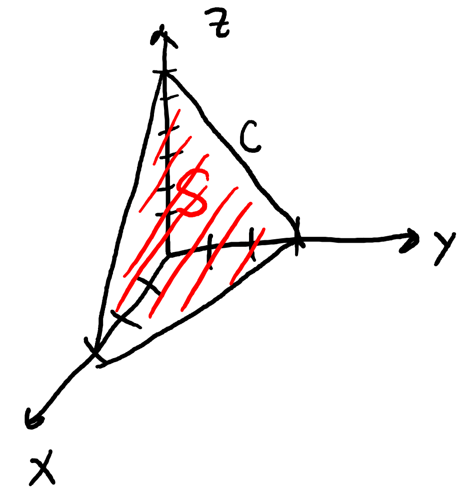
$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$$

$$= \iint_R \langle 0,0,1+2y\rangle \cdot \langle 0,1,1\rangle\, dA$$

$$= \iint_R (1+2y)\,dA = \int_0^{2\pi}\int_0^1 (1+2r\sin\theta)\,r\,dr\,d\theta$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = \langle 0,0,1+2y\rangle$$

Ex. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\mathbf{F} = -y^2\mathbf{i} + z\mathbf{j} + x\mathbf{k}$ and $C$ is the boundary of the portion of $2x + 2y + z = 6$ that lies in the first octant.



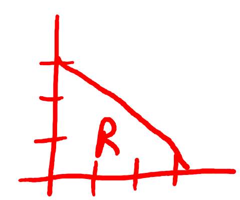

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$$

$$= \iint_R \langle -1, -1, 2y \rangle \cdot \langle 2, 2, 1 \rangle \, dA$$

$$= \int_0^3 \int_0^{3-x} (-2 - 2 + 2y) \, dA$$

$S: g(x,y) = 6 - 2x - 2y$

$R:$



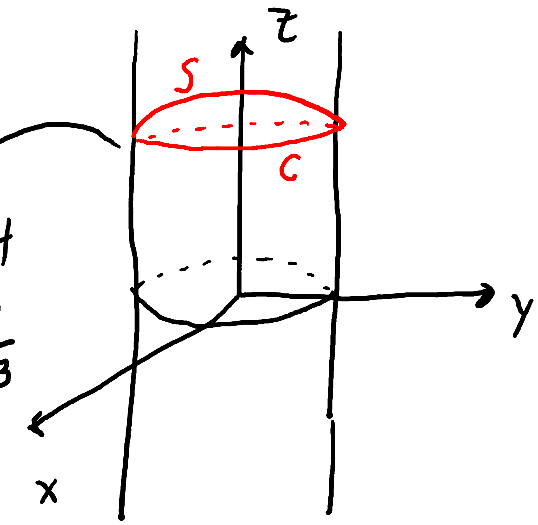

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & z & x \end{vmatrix} = \langle -1, -1, 2y \rangle$$

$\underline{\text{Ex.}}$ Evaluate $\iint\limits_S \operatorname{curl}\vec{F}\cdot d\vec{S}$ where $\mathbf{F} = xz\mathbf{i} + yz\mathbf{j} + xy\mathbf{k}$ and $S$ is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the $xy$-plane.

$$\iint\limits_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \int\limits_C \vec{F}\cdot d\vec{r}$$

$$= \int_0^{2\pi} \underbrace{\langle \sqrt{3}\cos t, \sqrt{3}\sin t, \sin t\cos t\rangle}_{\vec{F}} \cdot \underbrace{\langle -\sin t, \cos t, 0\rangle}_{d\vec{r}}\, dt$$

$$= \int_0^{2\pi} -\sqrt{3}\sin t\cos t + \sqrt{3}\sin t\cos t + 0\, dt = \boxed{0}$$




$1 + z^2 = 4$
$z^2 = 3$
$z = \sqrt{3}$

$C: \vec{r}(t) = \langle \cos t, \sin t, \sqrt{3}\rangle$
$0 \le t \le 2\pi$

# Line Integrals

| | |
|---|---|
| $f(x,y)$ | $\int_C f(x,y)\,ds = \int_a^b f(x,y)\sqrt{x'(t)^2 + y'(t)^2}\,dt$ |
| $\vec{F}(x,y)$ | $\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}'(t)\,dt$ |

If $\mathbf{F}(x,y)$ is conservative:

- $\int_C \vec{F}\cdot d\vec{r} = f(x(b),y(b)) - f(x(a),y(a))$
- We can change the path
- $\int_C \vec{F}\cdot d\vec{r} = 0$ if the path is closed

If $C$ is closed:

- $\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$ (Green's Theorem)

# Surface Integrals

| | $z=g(x,y)$ | $\vec{r}(u,v)$ |
|---|---|---|
| $\iint\limits_S f(x,y,z)\,dS$ | $\iint\limits_R f(x,y,g(x,y))\sqrt{1+{g_x}^2+{g_y}^2}\,dA$ | $\iint\limits_D f(x(u,v),y(u,v),z(u,v))\lvert\vec{r}_u\times\vec{r}_v\rvert\,dA$ |
| $\iint\limits_S \vec{F}\cdot\vec{n}\,dS$ | $\vec{n}\,dS=\langle -g_x,-g_y,1\rangle\,dA$ | $\vec{n}\,dS=(\vec{r}_u\times\vec{r}_v)\,dA$ |

If $S$ is closed:

- $\iint\limits_S \vec{F}\cdot d\vec{S}=\iiint\limits_E \operatorname{div}\vec{F}\,dV$ (Divergence Theorem)

If $C$ is the edge of $S$:

- $\int\limits_C \vec{F}\cdot d\vec{r}=\iint\limits_S \operatorname{curl}\vec{F}\cdot d\vec{S}$ (Stokes' Theorem)