Warm-up Problems

Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = \left\langle \frac{1}{2}xy, \frac{1}{4}x^2 \right\rangle$ and

1) C: $y = x^2$ from (0,0) to (1,1) $\quad \vec{r}_1 = \langle t, t^2 \rangle,\ 0 \le t \le 1$

2) C: $y = x$ from (0,0) to (1,1) $\quad \vec{r}_2 = \langle t, t \rangle,\ 0 \le t \le 1$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \left\langle \tfrac{1}{2}t\cdot t^2, \tfrac{1}{4}t^2 \right\rangle \cdot \langle 1, 2t\rangle\, dt = \int_0^1 \tfrac{1}{2}t^3 + \tfrac{1}{2}t^3\, dt = \int_0^1 t^3\, dt = \tfrac{1}{4}t^4 \Big|_0^1 = \boxed{\tfrac{1}{4}}$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \left\langle \tfrac{1}{2}t\cdot t, \tfrac{1}{4}t^2 \right\rangle \cdot \langle 1, 1\rangle\, dt = \int_0^1 \tfrac{1}{2}t^2 + \tfrac{1}{4}t^2\, dt = \int_0^1 \tfrac{3}{4}t^2\, dt$$

$$= \tfrac{1}{4}t^3 \Big|_0^1 = \boxed{\tfrac{1}{4}}$$

Fundamental Theorem for Line Integrals

Why did we get the same answer for both problems in the warm-up?

Wasn't the line integral supposed to depend on path?

What's so special about the vector field **F**?

Thm. The following are equivalent.

i. **F** is conservative.

ii. $\int_C \vec{F} \cdot d\vec{r}$ is independent of path.

iii. $\int_C \vec{F} \cdot d\vec{r} = 0$ for any closed curve C.

Fundamental Theorem of Line Integrals

Let **F** be a conservative vector field with potential function $f(x,y)$, and let C be given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ for $a \le t \le b$. Then

$$\int_C \vec{F} \cdot d\vec{r} = f\left(x(b), y(b)\right) - f\left(x(a), y(a)\right)$$

In other words, find the potential function and plug in the endpoints (just like FTOC).

Ex. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\mathbf{F} = 2xy\mathbf{i} + (x^2 + z^2)\mathbf{j} + 2yz\mathbf{k}$ and C is the piecewise smooth curve from (1,1,0) to (0,2,3).

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & x^2+z^2 & 2yz \end{vmatrix} = \langle 2z-2z, -(0-0), 2x-2x \rangle = \vec{0} \quad \therefore \vec{F} \text{ is cons.}$$

$f_x = 2xy$ $\quad f_y = x^2+z^2$ $\quad f_z = 2yz$

$f = x^2y$ $\quad f = x^2y + yz^2$ $\quad f = y^2z$ $\quad \rightarrow f = x^2y + yz^2$

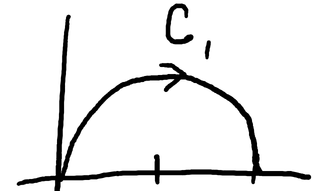
$$\int_C \vec{F}\cdot d\vec{r} = f(0,2,3) - f(1,1,0) = (0 + 2\cdot 3^2) - (1+0) = \boxed{17}$$

Ex. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\mathbf{F} = (y^3 + 1)\mathbf{i} + (3xy^2 + 1)\mathbf{j}$ and C is the semicircular path from (0,0) to (2,0). [Do this problem three ways.]

$(x-1)^2 + y^2 = 1$

$x^2 - 2x + 1 + y^2 = 1$

$y = \sqrt{2x - x^2}$



$\vec{r}_1 = \langle t, \sqrt{2t - t^2} \rangle$

$0 \le t \le 2$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^2 \langle (2t - t^2)^{3/2} + 1, 3t(2t - t^2) + 1 \rangle \cdot \langle 1, \tfrac{1}{2}(2t - t^2)^{-1/2}(2 - 2t) \rangle \, dt$$

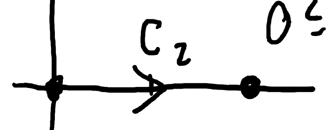
$P_y = 3y^2$

$Q_x = 3y^2$

$\therefore$ cons.

$\vec{r}_2 = \langle t, 0 \rangle$

$0 \le t \le 2$



$$\int_C \vec{F} \cdot d\vec{r} = \int_0^2 \langle 1, 1 \rangle \cdot \langle 1, 0 \rangle \, dt = \int_0^2 1 \, dt = t \Big|_0^2 = 2$$

$f_x = y^3 + 1$ $\quad f_y = 3xy^2 + 1$

$f = xy^3 + x$ $\quad f = xy^3 + y$

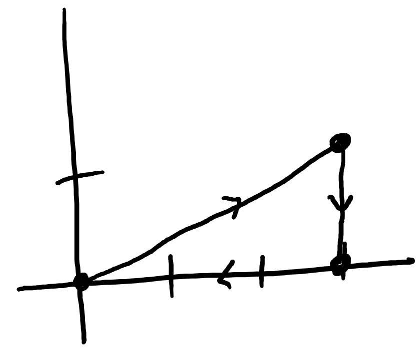
$f = xy^3 + x + y$

$$\int_C \vec{F} \cdot d\vec{r} = f(2,0) - f(0,0) = 2 - 0 = \boxed{2}$$

Ex. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\mathbf{F} = 2xy\mathbf{i} + (x^2 - y)\mathbf{j}$ and C is the path around the triangle with vertices (0,0), (3,1), and (3,0).

$P_y = 2x$

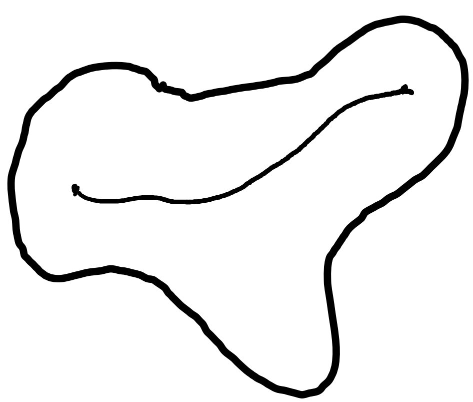
$Q_x = 2x$



$$\int_C \vec{F} \cdot d\vec{r} = 0$$

A region is <u>connected</u> if, between any two points, there is a piecewise smooth curve in the region that connects the points.

Yes



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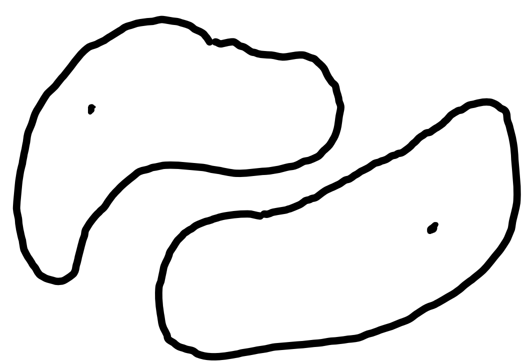
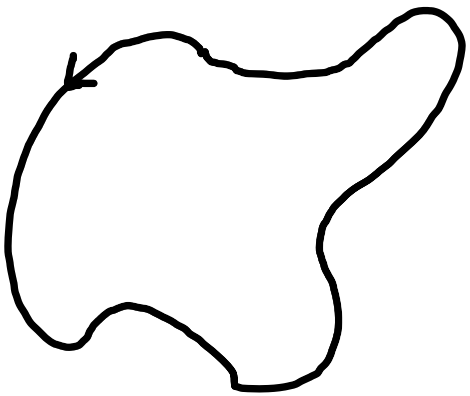


A curve is simple if it doesn't cross itself.

Yes



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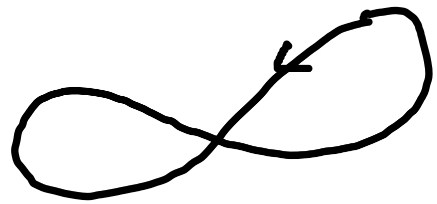
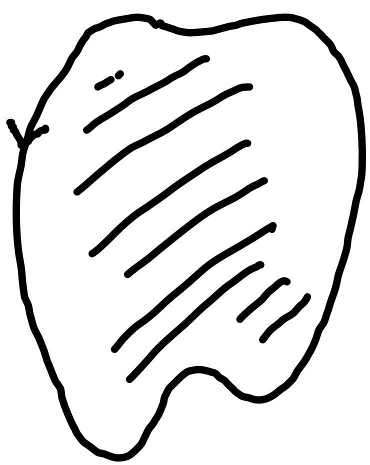


A region is simply connected if its boundary is one simple, closed curve.

Yes



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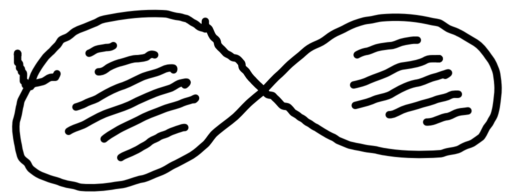
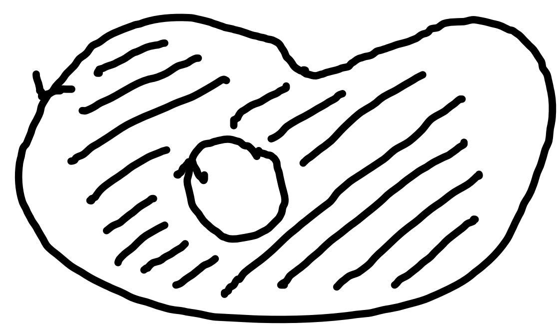
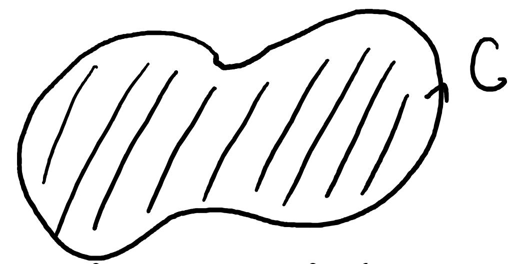






Green's Theorem

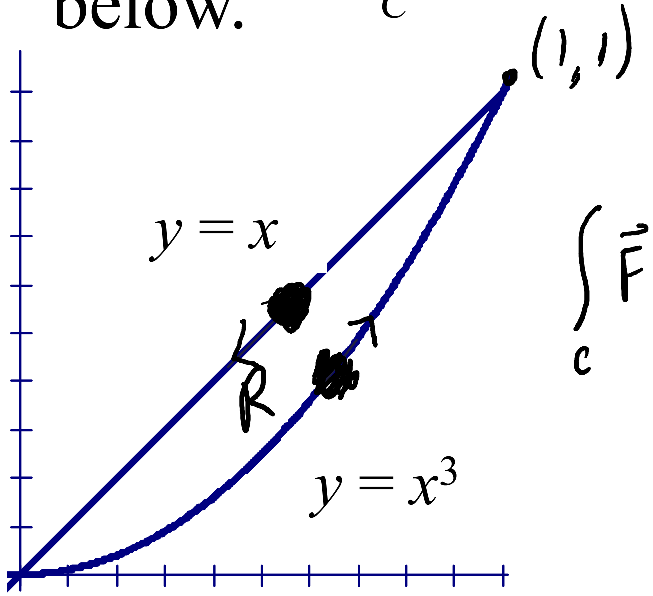
Let R be a simply connected region with a piecewise smooth boundary C, oriented counterclockwise. If M and N have continuous partial derivatives in an open region containing R, then

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

$$\oint_C P\,dx + Q\,dy \quad \text{or} \quad \circlearrowleft\!\!\!\!\!\!\oint_C P\,dx + Q\,dy$$

are sometimes used to show that C is a closed curve.

Ex. Evaluate $\int_C \overset{M}{y^3}\,dx + \overset{N}{\left(x^3 + 3xy^2\right)}\,dy$, where C is shown below.

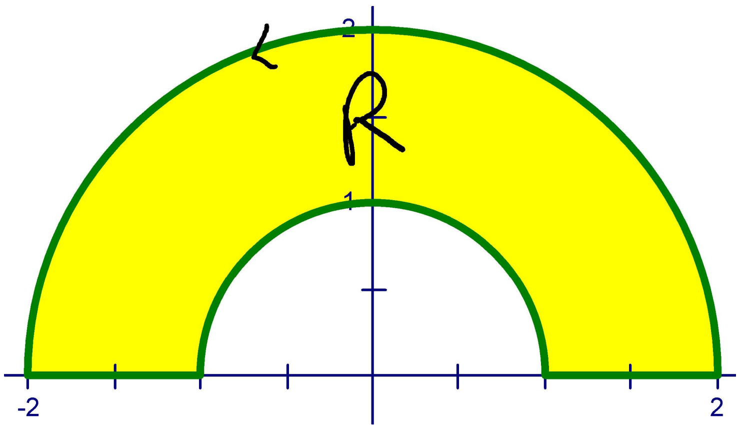


$$\int_C \vec{F}\cdot d\vec{r} = \iint_R \left(\left(3x^2+3y^2\right)-3y^2\right)dA$$

$$= \int_0^1 \int_{x^3}^{x} \left(3x^2\right)dy\,dx$$

⋮

Ex. Find the work done by $\mathbf{F} = y^2\mathbf{i} + 3xy\mathbf{j}$ on a particle travelling once around the boundary of the semiannular region shown below.

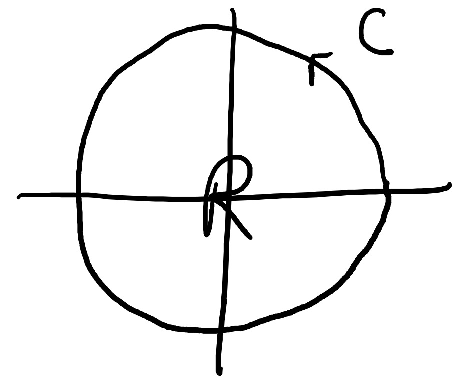


$$\int_C \vec{F}\cdot d\vec{r} = \iint_R 3y - 2y \, dA$$

$$= \int_0^{\pi}\int_1^2 r\sin\theta \cdot r\,dr\,d\theta$$

$\vdots$

Ex. Evaluate $\int_C y^3 dx + 3xy^2 dy$, where C is the boundary of the circle of radius 3.



$$\int_C \vec{F}\cdot d\vec{r} = \iint_R (3y^2 - 3y^2)\, dA = 0$$

It's possible to extend Green's Theorem to regions that have holes…

We just need to add a path.

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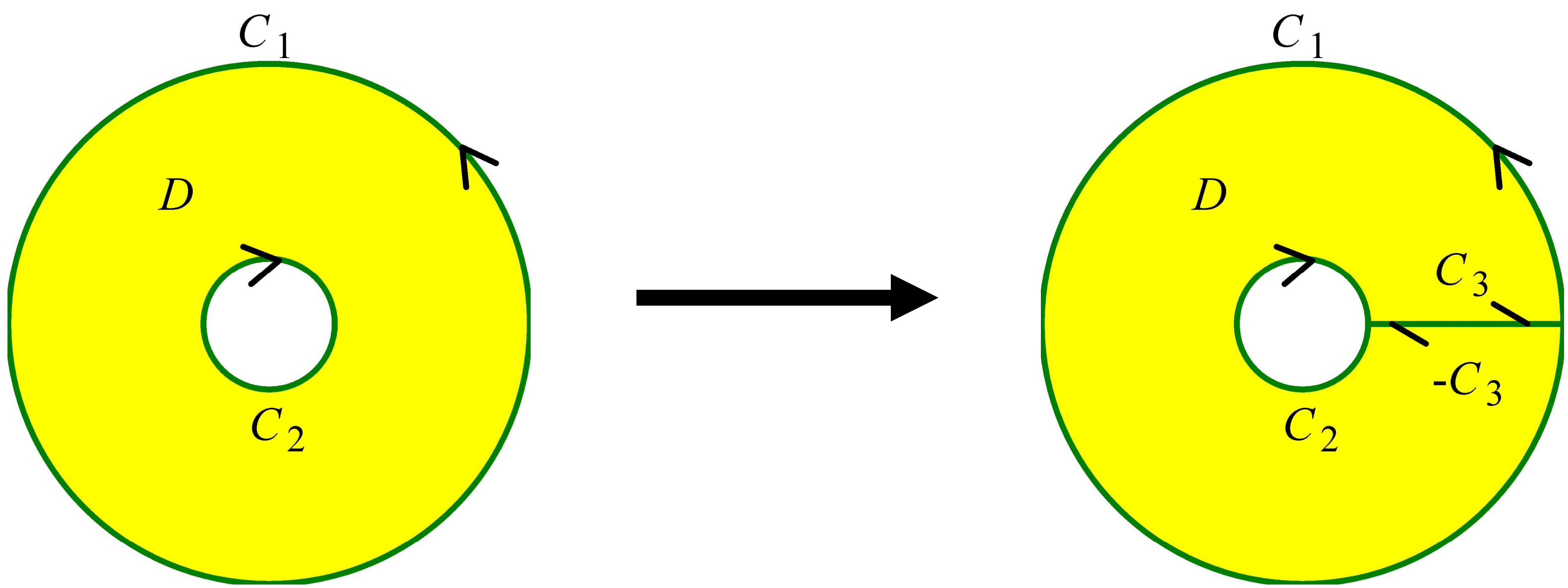

$C = C_1 + (-C_3) + C_2 + C_3$