Triple Integrals

<u>Ex.</u> Evaluate $\iiint_B xyz^2\,dV$ where

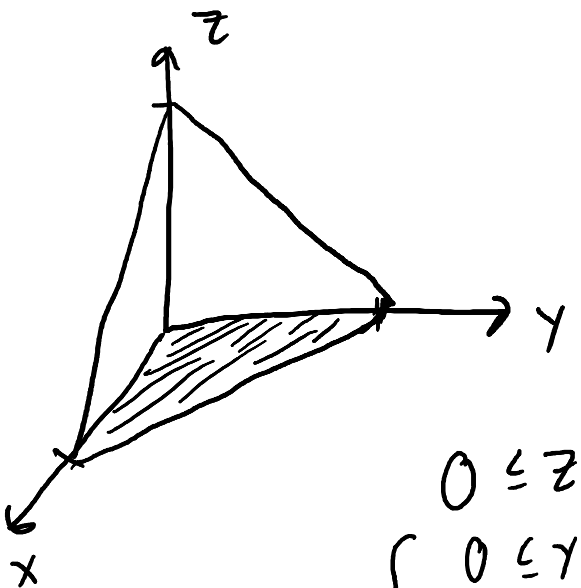
$$B = \{(x,y,z) \mid 0 \le x \le 1, -1 \le y \le 2, 0 \le z \le 3\}$$

$$= \int_0^1 \int_{-1}^2 \left[\int_0^3 xyz^2\,dz \right] dy\,dx = \int_0^1 \int_{-1}^2 \frac{1}{3}xyz^3 \Big|_0^3 dy\,dx = \int_0^1 \int_{-1}^2 9xy\,dy\,dx$$

$$= \int_0^1 \frac{9}{2}xy^2 \Big|_{-1}^2 dx = \int_0^1 18x - \frac{9}{2}x\,dx = \int_0^1 \frac{27}{2}x\,dx$$

$$= \frac{27}{4}x^2 \Big|_0^1 = \boxed{\frac{27}{4}}$$

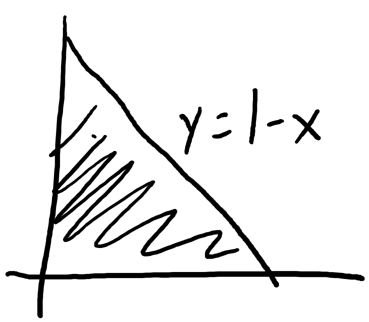
Ex. Evaluate $\iiint_E z\,dV$ where E is the solid bounded by the coordinate planes and the plane $x+y+z=1$. $\to z = 1-x-y$



$$\iiint_E z\,dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\,dz\,dy\,dx$$

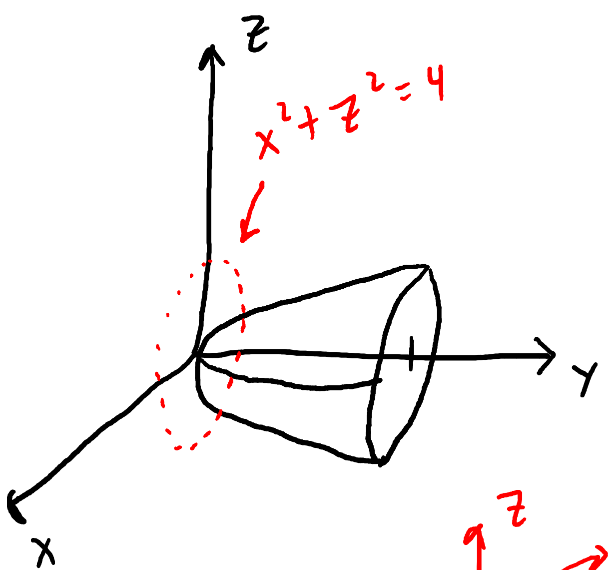
$$0 \le z \le 1-x-y$$

$$\begin{cases} 0 \le y \le 1-x \\ 0 \le x \le 1 \end{cases}$$



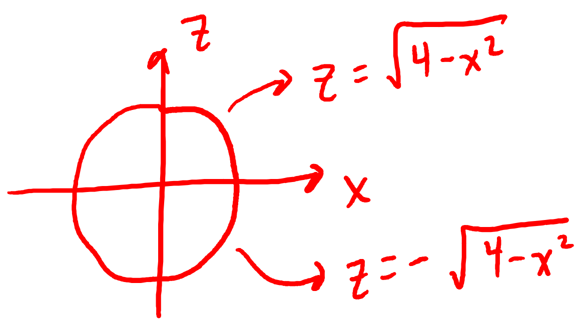
<u>Ex.</u> Evaluate $\iiint_Q \sqrt{x^2+z^2}\,dV$ where Q is the solid bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\; dy\, dz\, dx$$



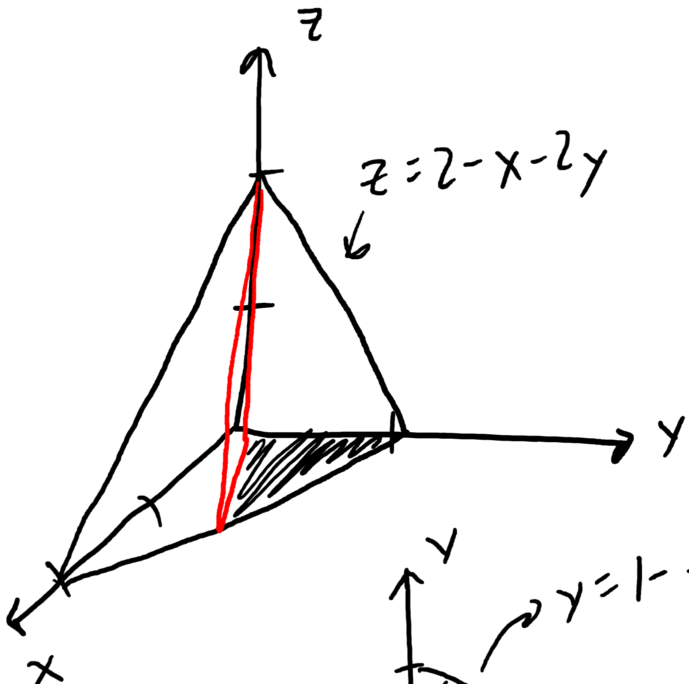
$x^2+z^2 \le y \le 4$

$$\begin{cases} -\sqrt{4-x^2} \le z \le \sqrt{4-x^2} \\ -2 \le x \le 2 \end{cases}$$



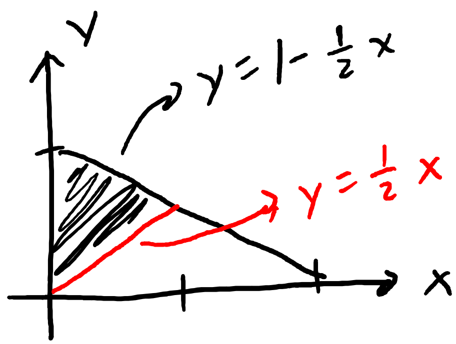
The volume of a solid Q is $V = \iiint_Q 1\,dV$

<u>Ex.</u> Use triple integrals to find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.



$$\int_0^1 \int_{\frac{1}{2}x}^{1-\frac{1}{2}x} \int_0^{2-x-2y} 1 \, dz\, dy\, dx$$

$$0 \le z \le 2 - x - 2y$$
$$\tfrac{1}{2}x \le y \le 1 - \tfrac{1}{2}x$$
$$0 \le x \le 1$$

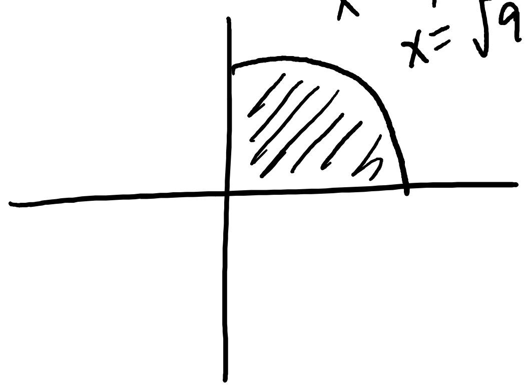


Ex. Rewrite $\displaystyle\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{6-x-y} dz\,dy\,dx$ to become $dz\,dx\,dy$.

$$\int_0^3 \int_0^{\sqrt{9-y^2}} \int_0^{6-x-y} dz\,dx\,dy$$

$0 \le y \le \sqrt{9-x^2}$

$0 \le x \le 3$



$0 \le x \le \sqrt{9-y^2}$

$0 \le y \le 3$

Triple Integrals in Spherical and Cylindrical

- In rectangular coordinates:

$$dV = dz\,dy\,dx$$

- In cylindrical coordinates:

$$dV = r\,dz\,dr\,d\theta$$

- In spherical coordinates:

$$dV = \rho^2 \sin\varphi\; d\rho\, d\varphi\, d\theta$$

Cylind → Rect

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

Rect → Cylind

$$x^2 + y^2 = r^2$$

$$\tan\theta = \tfrac{y}{x}$$

$$z = z$$

Spher → Rect

$$x = \rho\sin\varphi\cos\theta$$

$$y = \rho\sin\varphi\sin\theta$$

$$z = \rho\cos\varphi$$

Rect → Spher

$$x^2 + y^2 + z^2 = \rho^2$$

$$\tan\theta = \tfrac{y}{x}$$

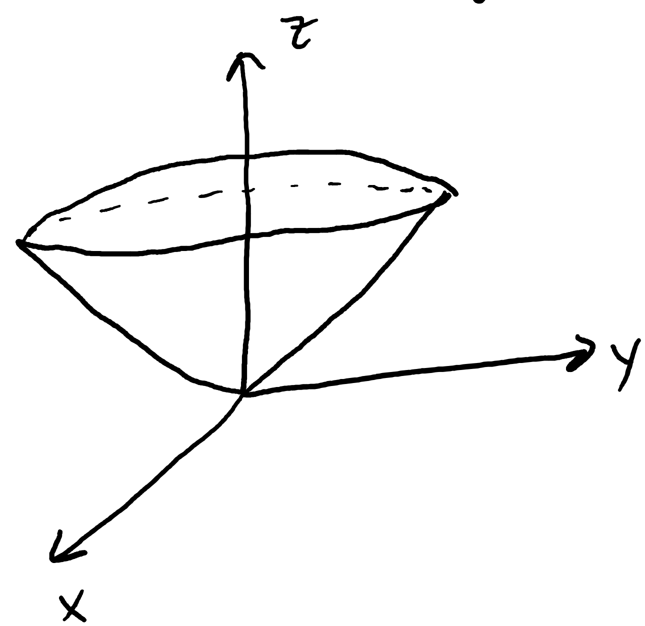
$$\cos\varphi = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

Ex. Sketch the solid whose volume is given by the integral $\int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \underbrace{\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta}_{dV} = \iiint_Q 1 \, dV$

$0 \le \theta \le 2\pi$

$0 \le \varphi \le \frac{\pi}{4}$

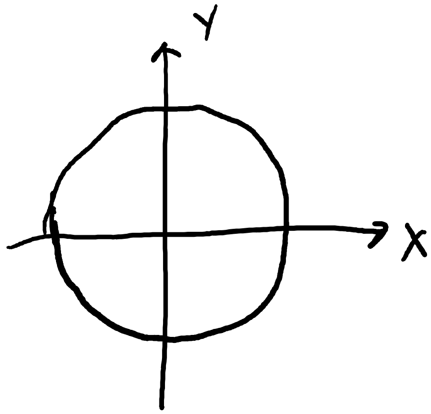
$0 \le \rho \le 1$

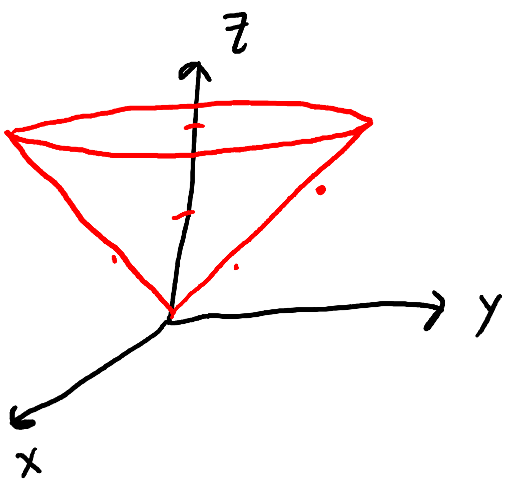


Ex. Evaluate $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}\left(x^2+y^2\right)dz\,dy\,dx = \int_0^{2\pi}\int_0^2\int_r^2 r^2\, r\, dz\, dr\, d\theta$

$z = 2$

$z = \sqrt{x^2+y^2} \rightarrow z = r$





$$= \int_0^{2\pi}\int_0^2 r^3 z\Big|_r^2\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 \left(2r^3 - r^4\right) dr\, d\theta$$

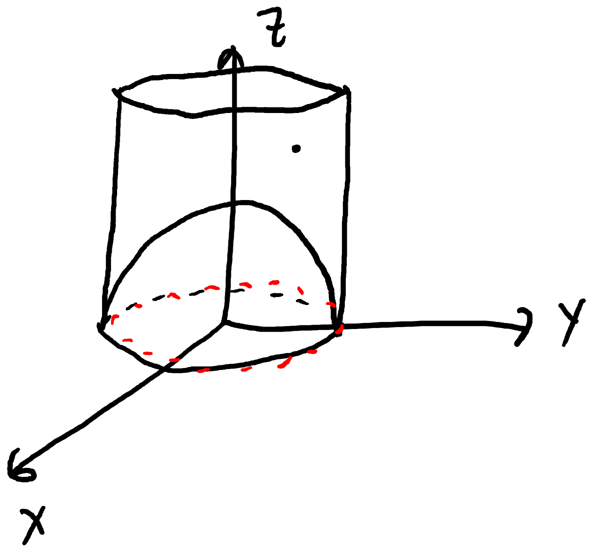
$$\vdots$$

Ex. A solid lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$. The density at a point is proportional to its distance to the z-axis. Find the mass.

$1 - (x^2 + y^2)$

$1 - r^2$

$$m = \iiint_Q \rho \, dV$$



$$= \underbrace{\int_0^{2\pi} \int_0^{1}} \int_{1-r^2}^{4} k r \cdot r \, dz \, dr \, d\theta$$

$\rho = kr$

Ex. Evaluate $\iiint\limits_B e^{\left(\underbrace{x^2+y^2+z^2}_{\rho^2}\right)^{3/2}}\,dV$, where B is the unit ball.

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} e^{\rho^3}\,\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

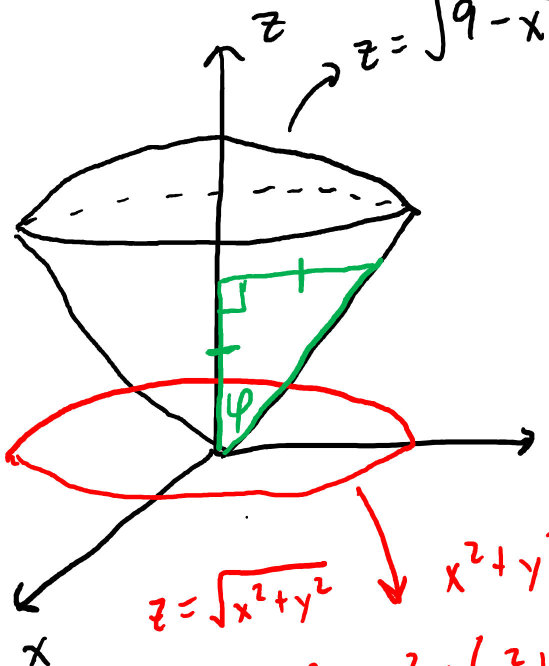
$$= \int_0^{2\pi}\int_0^{\pi} \frac{1}{3}e^{\rho^3}\Big|_0^1 \sin\varphi\, d\varphi\, d\theta = \int_0^{2\pi}\int_0^{\pi} \frac{1}{3}(e-1)\sin\varphi\, d\varphi\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{3}(e-1)\,{-\cos\varphi}\Big|_0^{\pi} d\theta = \int_0^{2\pi} \frac{1}{3}(e-1)(-\cos\pi+\cos 0)\,d\theta$$

$$= \int_0^{2\pi} \frac{2}{3}(e-1)\,d\theta = \frac{2}{3}(e-1)\,\theta\Big|_0^{2\pi} = \boxed{\frac{4\pi}{3}(e-1)}$$

<u>Ex.</u> Find the volume of the solid that lies above $z = \sqrt{x^2 + y^2}$ and below $x^2 + y^2 + z^2 = 9$.

$z = \sqrt{9 - x^2 - y^2}$



$$x^2 + y^2 + z^2 = 9$$
$$x^2 + y^2 + (x^2 + y^2) = 9$$
$$2(x^2 + y^2) = 9$$
$$x^2 + y^2 = \frac{9}{2}$$
$$r = \frac{3}{\sqrt{2}}$$

$$V = \iiint_Q 1\, dV$$

$$= \int_{-\frac{3}{\sqrt{2}}}^{\frac{3}{\sqrt{2}}} \int_{-\sqrt{\frac{9}{2} - x^2}}^{\sqrt{\frac{9}{2} - x^2}} \int_{\sqrt{x^2 + y^2}}^{\sqrt{9 - x^2 - y^2}} dz\, dy\, dx$$

$$= \int_0^{2\pi} \int_0^{3/\sqrt{2}} \int_r^{\sqrt{9 - r^2}} r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \int_0^3 \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$