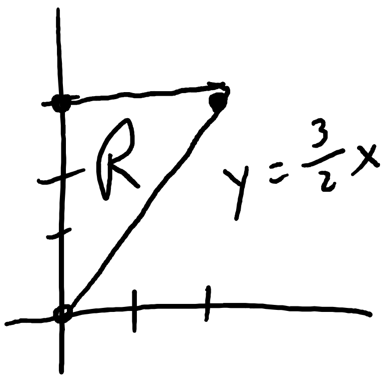
Applying Multiple Integrals

Thm. Let $\rho(x,y)$ be the density of a lamina corresponding to a plane region R. The mass of the lamina is

$$m = \iint_R \rho(x, y)\, dA$$

Ex. Find the mass of the triangular lamina with vertices $(0,0)$, $(0,3)$, and $(2,3)$ if the density is given by $\rho = 2x + y$.



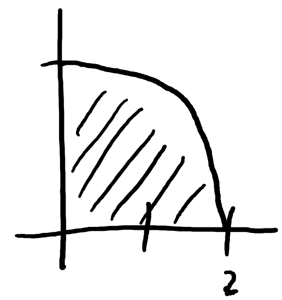
$$m = \iint_R (2x+y)\,dA$$

$$= \int_0^2 \int_{\frac{3}{2}x}^{3} (2x+y)\,dy\,dx$$

$$\vdots$$

Ex. Find the mass of the lamina corresponding to the first quadrant portion of $x^2 + y^2 = 4$ if $\rho = k\sqrt{x^2 + y^2}$

$$m = \iint_R k\sqrt{x^2+y^2}\, dA$$



$$= \int_0^{\pi/2}\int_0^2 kr \cdot r\, dr\, d\theta$$

$\vdots$

<u>Thm.</u> Let $\rho(x,y)$ be the density of a planar lamina R. The <u>moments of mass</u> about the x-axis and y-axis are

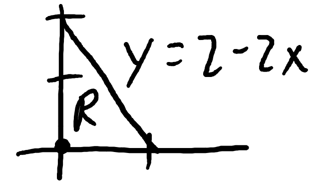
$$M_x = \iint_R y\rho(x,y)\,dA \qquad M_y = \iint_R x\rho(x,y)\,dA$$

The <u>center of mass</u> is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right)$$

$$= \left(\frac{\iint_R x\rho\,dA}{m}, \frac{\iint_R y\rho\,dA}{m}\right)$$

Ex. Find the mass and center of mass of the triangular lamina with vertices (0,0), (1,0), and (0,2) if the density is $\rho = 1 + 3x + y$.



$$m = \iint_R (1+3x+y)\,dA = \int_0^1 \int_0^{2-2x} (1+3x+y)\,dy\,dx = \int_0^1 y + 3xy + \tfrac{1}{2}y^2 \Big|_0^{2-2x} dx$$

$$= \int_0^1 (2-2x) + 3x(2-2x) + \tfrac{1}{2}(2-2x)^2\,dx = \int_0^1 2-2x+6x-6x^2+2-4x+2x^2\,dx = \int_0^1 (-4x^2+4)\,dx$$

$$= -\tfrac{4}{3}x^3 + 4x \Big|_0^1 = -\tfrac{4}{3} + 4 = \tfrac{8}{3}$$

$$M_y = \iint_R x(1+3x+y)\,dA = \int_0^1 \int_0^{2-2x} (x+3x^2+xy)\,dy\,dx = \int_0^1 xy + 3x^2y + \tfrac{1}{2}xy^2 \Big|_0^{2-2x} dx$$

$$= \int_0^1 x(2-2x) + 3x^2(2-2x) + \tfrac{1}{2}x(2-2x)^2\,dx = \int_0^1 2x-2x^2+6x^2-6x^3+2x-4x^2+2x^3\,dx$$

$$= \int_0^1 (-4x^3+4x)\,dx = -x^4 + 2x^2 \Big|_0^1 = -1 + 2 = 1$$

$$M_x = \iint_R y(1+3x+y)\,dA = \int_0^1 \int_0^{2-2x} (y+3xy+y^2)\,dy\,dx = \int_0^1 \frac{1}{2}y^2 + \frac{3}{2}xy^2 + \frac{1}{3}y^3 \Big|_0^{2-2x} dx$$

$$= \int_0^1 \frac{1}{2}(2-2x)^2 + \frac{3}{2}x(2-2x)^2 + \frac{1}{3}(2-2x)^3\,dx$$

$$= \int_0^1 \frac{1}{2}(4-8x+4x^2) + \frac{3}{2}x(4-8x+4x^2) + \frac{1}{3}(8-24x+24x^2-8x^3)\,dx$$

$$= \int_0^1 (2 - 4x + 2x^2 + 6x - 12x^2 + 6x^3 + \frac{8}{3} - 8x + 8x^2 - \frac{8}{3}x^3\,dx$$

$$= \int_0^1 \frac{10}{3}x^3 - 2x^2 - 6x + \frac{14}{3}\,dx = \frac{10}{12}x^4 - \frac{2}{3}x^3 - 3x^2 + \frac{14}{3}x \Big|_0^1$$

$$= \frac{5}{6} - \frac{2}{3} - 3 + \frac{14}{3} = \frac{5-4-18+28}{6} = \frac{11}{6}$$

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \left(\frac{1}{8/3}, \frac{11/6}{8/3}\right) = \boxed{\left(\frac{3}{8}, \frac{11}{16}\right)}$$

Thm. Let $\rho(x,y)$ be the density of a planar lamina R. The moment of inertia about the x-axis and y-axis are

$$I_x = \iint_R y^2 \rho(x,y)\,dA \qquad I_y = \iint_R x^2 \rho(x,y)\,dA$$

The moment of inertia about the origin is

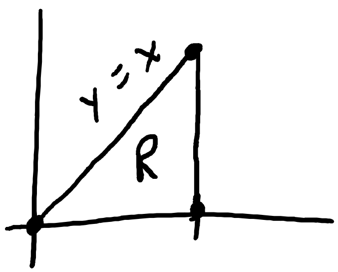
$$I_O = \iint_R \left(x^2 + y^2\right)\rho(x,y)\,dA$$

Please note that $I_x + I_y = I_O$

Thm. The surface area of $z = f(x,y)$ over the closed region R is given by

$$S = \iint_R \sqrt{1 + f_x^{\,2} + f_y^{\,2}}\, dA$$

<u>Ex.</u> Find the surface area of the part of the surface $z = x^2 + 2y$ that lies above the triangular region with vertices (0,0), (1,0), and (1,1).



$$S = \iint_R \sqrt{1 + (2x)^2 + (2)^2}\, dA$$

$$= \int_0^1 \int_0^x \sqrt{4x^2 + 5}\, dy\, dx = \int_0^1 y\sqrt{4x^2+5}\,\Big|_0^x\, dx$$

$$= \int_0^1 x\sqrt{4x^2+5}\, dx = \int_{*}^{**} u^{1/2} \cdot \frac{1}{8}\, du = \frac{1}{8}\cdot\frac{2}{3}\, u^{3/2}\Big|_{*}^{**}$$

$$= \frac{1}{12}\left(4x^2+5\right)^{3/2}\Big|_0^1 = \boxed{\frac{1}{12}\left(9^{3/2} - 5^{3/2}\right)}$$

$$\boxed{\begin{aligned} u &= 4x^2 + 5 \\ du &= 8x\,dx \\ \tfrac{1}{8}\,du &= x\,dx \end{aligned}}$$

Ex. Find the surface area of the part of the surface $z = 1 + x^2 + y^2$ that lies above the unit circle.

$$S = \iint_R \sqrt{1 + \underbrace{(2x)^2 + (2y)^2}_{4(x^2+y^2)}}\, dA = \int_0^{2\pi}\int_0^1 \sqrt{1+4r^2}\, r\, dr\, d\theta$$

$\vdots$

$u = 1 + 4r^2$

Ex. Set up a double integral that gives the surface area of the part of the $f(x,y) = x^2 - 3xy - y^2$ that lies over $R = \{(x,y) | 0 \le x \le 4, 0 \le y \le x\}$

$$S = \int_0^4 \int_0^x \sqrt{1 + (2x-3y)^2 + (-3x-2y)^2}\, dy\, dx$$