Multiple Integrals

<u>Ex.</u> $\int_1^x \left(2x^2y^{-2}+2y\right)dy = -2x^2y^{-1}+y^2 \Big|_1^x$

$= \left(-2x^2x^{-1}+x^2\right)-\left(-2x^2+1\right)$

$= -2x+x^2+2x^2-1$

Ex. $\displaystyle\int_{-1}^{1}\left[\int_{-2}^{2}\left(x^2-y^2\right)dy\right]dx = \int_{-1}^{1} x^2y-\frac{1}{3}y^3\Big|_{-2}^{2}\,dx$

$$= \int_{-1}^{1}\left(2x^2-\frac{8}{3}\right)-\left(-2x^2+\frac{8}{3}\right)dy = \int_{-1}^{1}\left(4x^2-\frac{16}{3}\right)dx$$

$$= \frac{4}{3}x^3-\frac{16}{3}x\Big|_{-1}^{1} = \left(\frac{4}{3}-\frac{16}{3}\right)-\left(\frac{-4}{3}+\frac{16}{3}\right)$$

$$= \frac{-12}{3}-\frac{12}{3} = \frac{-24}{3} = \boxed{-8}$$

The bounds of the integral represent a region in the *xy*-plane, and the function represents a height

→ So the integral represents the volume under the function $f(x,y)$ that lies above the region R in the *xy*-plane

$$\iint_R f(x,y)\,dA$$ → General double integral

$$\int_c^d \int_a^b f(x,y)\,dy\,dx$$ → Iterated integral

Ex. $\int_0^3 \left[\int_1^2 x^2 y \, dy \right] dx \quad = \int_0^3 \frac{1}{2} x^2 y^2 \Big|_1^2 \, dx$

$= \int_0^3 \frac{1}{2} x^2 \cdot 2^2 - \frac{1}{2} x^2 \cdot 1^2 \, dx = \int_0^3 2x^2 - \frac{1}{2} x^2 \, dx = \int_0^3 \frac{3}{2} x^2 \, dx$

$= \frac{1}{2} x^3 \Big|_0^3 = \frac{27}{2}$

<u>Ex.</u> $\int_1^2 \int_0^3 x^2 y \, dx dy = \int_1^2 \frac{1}{3}x^3 y \Big|_0^3 dy = \int_1^2 9y \, dy$

$= \frac{9}{2} y^2 \Big|_1^2 = \frac{9}{2} \cdot 4 - \frac{9}{2} \cdot 1 = \frac{36}{2} - \frac{9}{2} = \frac{27}{2}$

x and y have constant endpts.

When R is rectangular, order of integration doesn't matter.

Ex. Evaluate $\iint_R \left(x-3y^2\right)dA$, where

$$R=\left\{(x,y)\,\middle|\,0\le x\le 2, 1\le y\le 2\right\}$$

$$=\int_0^2\int_1^2 \left(x-3y^2\right)dy\,dx = \int_0^2 xy - y^3\Big|_1^2\,dx$$

$$=\int_0^2 (2x-8)-(x-1)\,dx = \int_0^2 (x-7)\,dx = \frac{1}{2}x^2 - 7x\Big|_0^2$$

$$= 2-14 = \boxed{-12}$$

Ex. Evaluate $\iint_R y\sin(xy)\,dA$, where

$$R=[1,2]\times[0,\pi]$$

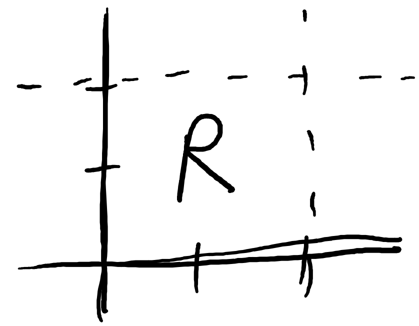
x y

$$\int_0^{\pi}\int_1^{2} y\sin(xy)\,dx\,dy = \int_0^{\pi} -\cancel{y}\cdot\frac{1}{\cancel{y}}\cos(xy)\Big|_1^2\,dy$$

$$= \int_0^{\pi}\left(-\cos(2y)+\cos y\right)dy = -\frac{1}{2}\sin 2y + \sin y\Big|_0^{\pi} = 0$$

Ex. Find the volume of the solid bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the coordinate planes.

$\hookrightarrow z = 16 - x^2 - 2y^2$



$$\iint_R \left(16 - x^2 - 2y^2\right) dA$$

$$= \int_0^2 \int_0^2 \left(16 - x^2 - 2y^2\right) dy\, dx$$

<u>Ex.</u> Find the average value of $f(x,y) = \sin x \cos y$ over the region $R = [0,\frac{\pi}{2}] \times [0,\frac{\pi}{2}]$.

$$\frac{1}{\text{area of } R} \iint_R f(x,y)\,dA = \frac{1}{\frac{\pi}{2}\cdot\frac{\pi}{2}} \int_0^{\pi/2}\int_0^{\pi/2} \sin x \cos y \, dy\, dx$$

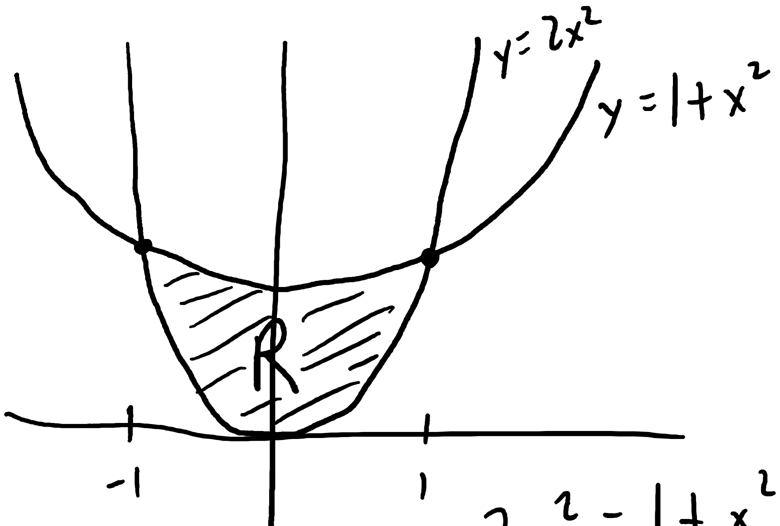
$\vdots$

$f(x)$ on $[a,b]$

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

To evaluate a double integral over a non-rectangular region, we need to describe the region as boundaries of x and y.

Ex. Describe the region R bounded by $y = 2x^2$ and $y = 1 + x^2$.



$$\boxed{\begin{array}{l} 2x^2 \le y \le 1 + x^2 \\ -1 \le x \le 1 \end{array}}$$

$$2x^2 = 1 + x^2$$
$$x^2 = 1$$
$$x = \pm 1$$

Ex. Evaluate $\iint_R (x+2y)\,dA$, where R is the region from the previous example.

$$= \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x+2y)\,dy\,dx = \int_{-1}^{1} xy + y^2 \Big|_{2x^2}^{1+x^2} dx$$

$$= \int_{-1}^{1} \left[x(1+x^2) + (1+x^2)^2\right] - \left[x \cdot 2x^2 + (2x^2)^2\right] dx$$

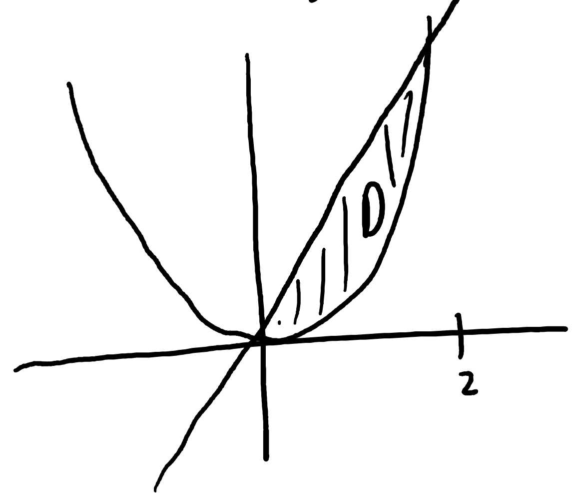
$$\vdots$$

Ex. Find the volume that lies below $z = x^2 + y^2$ and above the region D in the xy-plane bounded by $y = 2x$ and $y = x^2$.

$$V = \iint_D (x^2+y^2)\,dA$$

$x^2 \le y \le 2x$

$0 \le x \le 2$



$$= \int_0^2 \int_{x^2}^{2x} (x^2+y^2)\,dy\,dx$$

$$= \int_0^2 x^2 y + \frac{1}{3} y^3 \Big|_{x^2}^{2x} dx = \int_0^2 \Big(\overbrace{2x^3+\frac{8}{3}x^3}^{\frac{14}{3}x^3}\Big) - \Big(x^4+\frac{1}{3}x^6\Big)\,dx$$

$$= \frac{14}{12}x^4 - \frac{1}{5}x^5 - \frac{1}{21}x^7 \Big|_0^2 = \boxed{\frac{7}{6}(16) - \frac{32}{5} - \frac{128}{21}}$$

$x^2 = 2x$

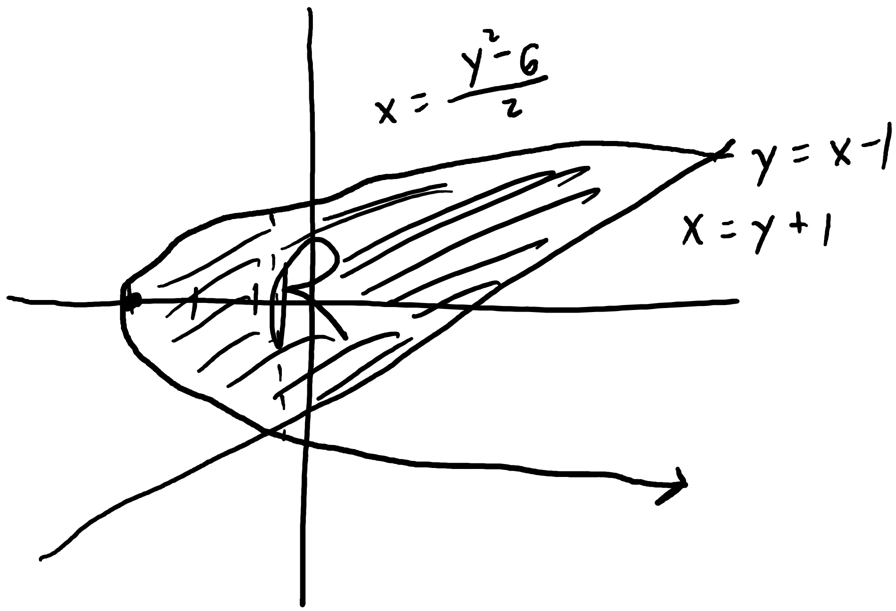
$x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0 \quad x = 2$

Ex. Evaluate $\iint_R xy\,dA$, where R is the region bounded by $y = x - 1$ and $y^2 = 2x + 6$.

$$\int_{-2}^{4} \int_{\frac{y^2-6}{2}}^{y+1} xy \, dx\, dy$$



$$\frac{y^2-6}{2} \le x \le y+1$$

$$-2 \le y \le 4$$

$$\frac{y^2-6}{2} = y+1$$

$$y^2-6 = 2y+2$$

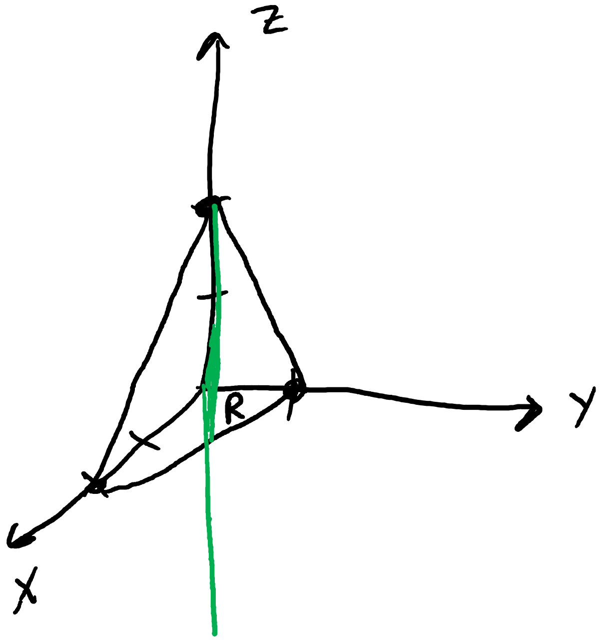
$$y^2-2y-8=0$$

$$(y-4)(y+2)=0$$

$$y = 4, -2$$

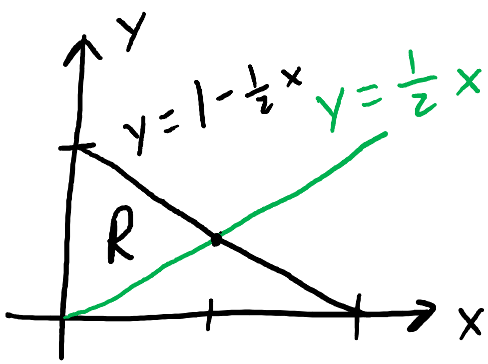
Ex. Find the volume bounded by $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$

$z = 2 - x - 2y$ $\quad$ $y = \frac{1}{2}x$



$$V = \iint_R (2 - x - 2y)\,dA$$

$$= \int_0^1 \int_{\frac{1}{2}x}^{1-\frac{1}{2}x} (2 - x - 2y)\,dy\,dx$$



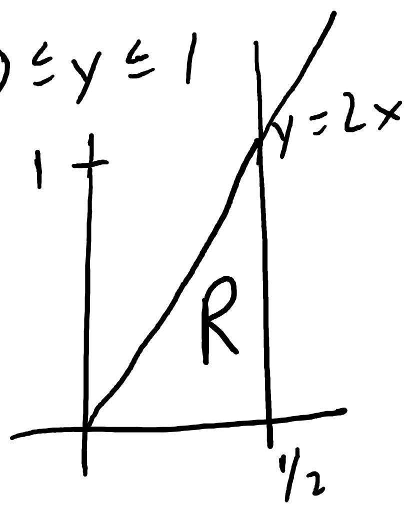
$\frac{1}{2}x \le y \le 1 - \frac{1}{2}x$

$0 \le x \le 1$

Ex. Evaluate $\int_0^1 \int_{y/2}^{1/2} e^{-x^2}\,dx\,dy$ by first switching the order of integration.

$\frac{y}{2} \le x \le \frac{1}{2}$

$0 \le y \le 1$



$0 \le y \le 2x$

$0 \le x \le \frac{1}{2}$

$$= \int_0^{1/2} \int_0^{2x} e^{-x^2}\,dy\,dx$$

$$= \int_0^{1/2} y e^{-x^2} \Big|_0^{2x} dx$$

$$= \int_0^{1/2} 2x e^{-x^2}\,dx = -e^{-x^2} \Big|_0^{1/2}$$

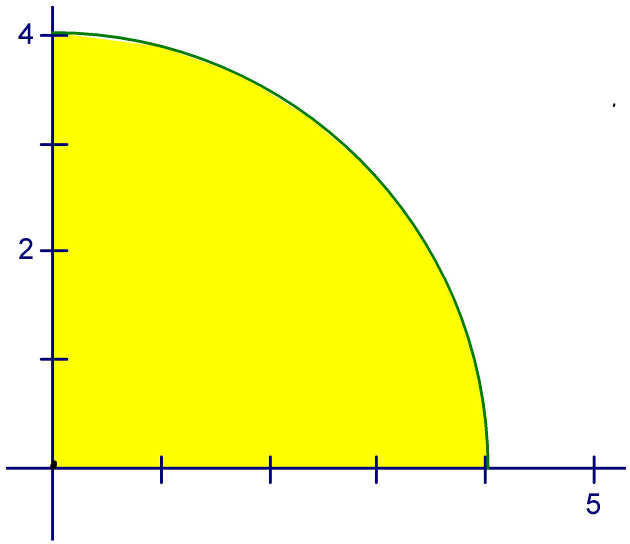
$u = -x^2$

$$= -e^{-1/4} + 1$$

Double Integrals in Polar

When evaluating $\iint\limits_R f(x,y)\,dA$, it may be easier to describe R using polar coordinates.

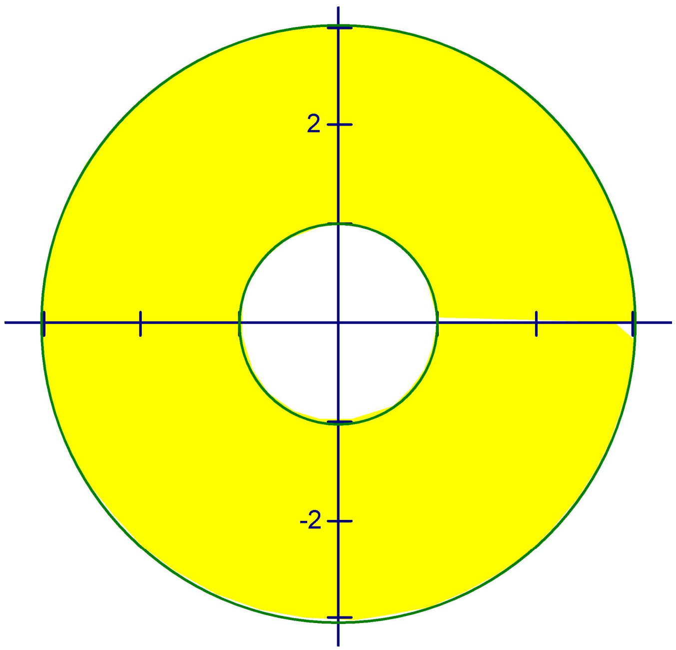
Ex. Describe the region using polar coordinates.



$0 \le r \le 4$

$0 \le \theta \le \frac{\pi}{2}$

Ex. Describe the region using polar coordinates.



$1 \le r \le 3$

$0 \le \theta \le 2\pi$

- In rectangular coordinates:

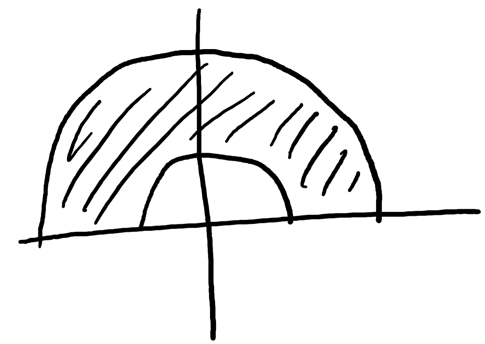
$$dA = dy\,dx$$

- In polar coordinates:

$$dA = r\,dr\,d\theta$$

Ex. Evaluate $\iint\limits_R (3x+4y^2)\,dA$ where R is the region in the first two quadrants bounded by $x^2+y^2=1$ and $x^2+y^2=4$.

$$\int_0^{\pi}\int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\,r\,dr\,d\theta$$



$$=\int_0^{\pi}\int_1^2 (3r^2\cos\theta + 4r^3\sin^2\theta)\,dr\,d\theta$$

$1 \le r \le 2$

$0 \le \theta \le \pi$

$$=\int_0^{\pi} r^3\cos\theta + r^4\sin^2\theta \Big|_1^2\,d\theta$$

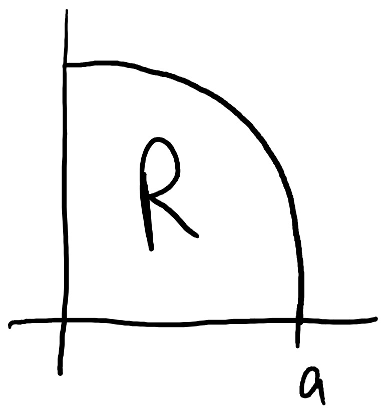
$$=\int_0^{\pi} (8\cos\theta + 16\sin^2\theta) - (\cos\theta + \sin^2\theta)\,d\theta = \int_0^{\pi} (7\cos\theta + 15\sin^2\theta)\,d\theta$$

$$=\int_0^{\pi} 7\cos\theta + \tfrac{15}{2}(1-\cos 2\theta)\,d\theta = 7\sin\theta + \tfrac{15}{2}\theta - \tfrac{15}{4}\sin 2\theta\Big|_0^{\pi} = \boxed{\frac{15\pi}{2}}$$

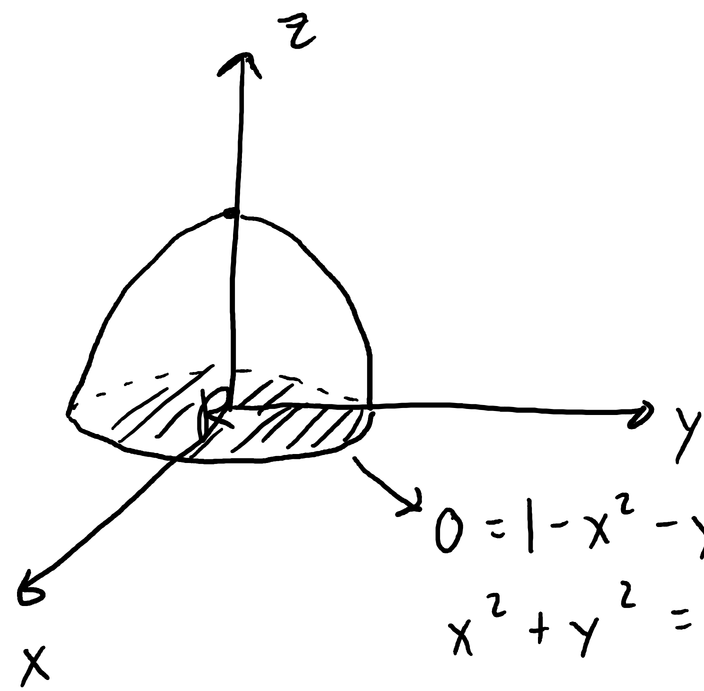
__Ex.__ Evaluate $\underbrace{\int_0^a \int_0^{\sqrt{a^2-x^2}}}_{R} x\,\underbrace{dy\,dx} = \int_0^{\pi/2}\int_0^a r\cos\theta \; r\,dr\,d\theta$

$\vdots$

$y = \sqrt{a^2 - x^2}$



Ex. Find the volume of the solid bounded by $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.



$0 = 1 - x^2 - y^2$

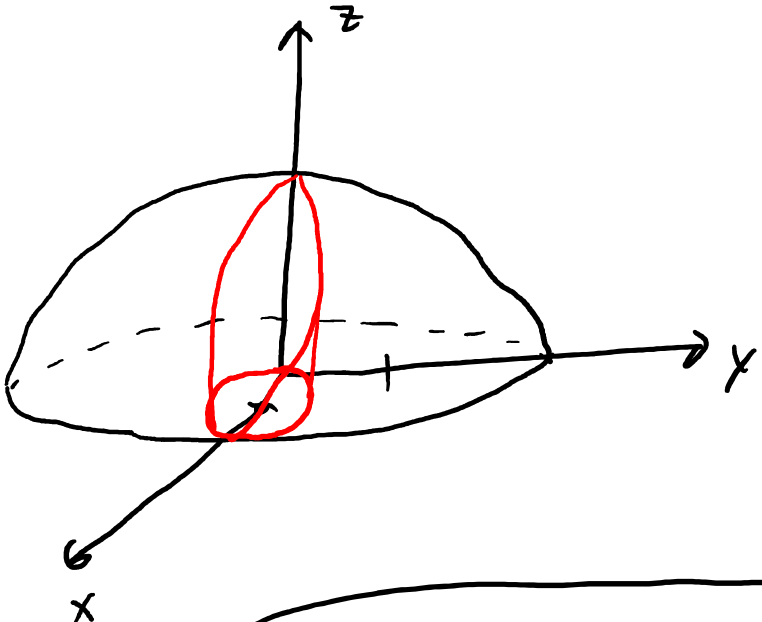
$x^2 + y^2 = 1$

$$V = \iint_R (1 - x^2 - y^2)\, dA$$

$$= \int_0^{2\pi} \int_0^1 (1 - r^2)\, r\, dr\, d\theta$$

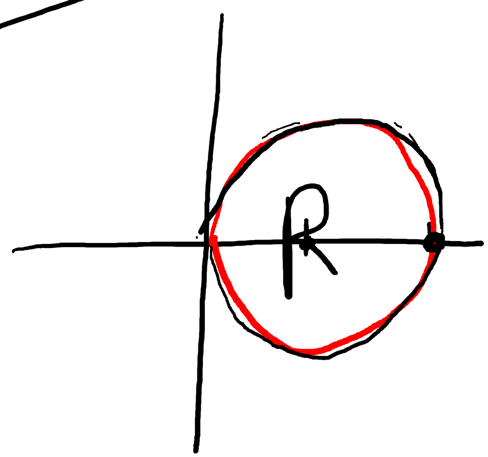
$\vdots$

Ex. Find the volume that lies under $z = \sqrt{4 - x^2 - y^2}$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.



$$V = \iint_R \sqrt{4 - x^2 - y^2}\, dA$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} \sqrt{4 - r^2}\; r\, dr\, d\theta$$



$x^2 + y^2 = 2x \longrightarrow r^2 = 2r\cos\theta$

$r = 2\cos\theta$

$x^2 - 2x + \underline{1} + y^2 = \underline{1}$

$(x-1)^2 + y^2 = 1$

$0 \le r \le 2\cos\theta$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$\int_{-\pi/2}^{\pi/2}\int_{0}^{2\cos\theta} \sqrt{4-r^2}\; r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} -\frac{1}{3}\left(4-r^2\right)^{3/2}\Big|_{0}^{2\cos\theta} d\theta$$

$$\boxed{u = 4-r^2}$$

$$= \int_{-\pi/2}^{\pi/2} -\frac{1}{3}\left(4-4\cos^2\theta\right)^{3/2} + \frac{1}{3}4^{3/2}\, d\theta = \int_{-\pi/2}^{\pi/2} -\frac{1}{3}\left(4\sin^2\theta\right)^{3/2} + \frac{8}{3}\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\left(-\frac{8}{3}\sin^3\theta + \frac{8}{3}\right)d\theta =$$

$$\sin^2\theta \sin\theta$$

$$\left(1-\cos^2\theta\right)\sin\theta$$

$$\left(1-u^2\right)(-1)\,du$$

$$\boxed{\begin{aligned} u &= \cos\theta \\ du &= -\sin\theta\, d\theta \\ -du &= \sin\theta\, d\theta \end{aligned}}$$