Optimization

Ex. A rectangular box lies on the xy-~~axis~~ plane with one vertex on the origin. If the opposite vertex lies on the plane $3x + 2y + z = 12$, find the maximum volume.

$$z = 12 - 3x - 2y$$

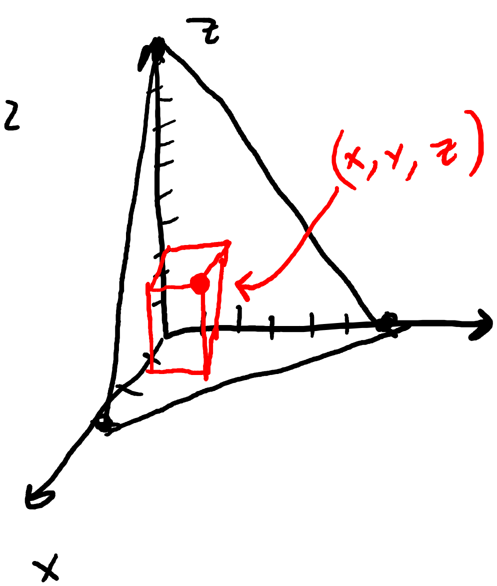
$$V = xyz = xy(12 - 3x - 2y) = 12xy - 3x^2y - 2xy^2$$

$$V_x = 12y - 6xy - 2y^2$$
$$= 2y(6 - 3x - y) = 0$$
$$3x + y = 6$$

$$V_y = 12x - 3x^2 - 4xy$$
$$= x(12 - 3x - 4y) = 0$$
$$3x + 4y = 12$$
$$3x + y = 6$$
$$3y = 6$$
$$y = 2$$

$$3x + 2 = 6$$
$$3x = 4$$
$$x = \frac{4}{3}$$

$$z = 12 - 3\left(\frac{4}{3}\right) - 2(2)$$
$$= 12 - 4 - 4 = 4$$



$$V = \frac{4}{3}(2)(4) = \frac{32}{3}$$

Ex. A manufacturer determined that the profit, P, obtained by producing x DVD players and y cassette players is given by

$$P(x,y) = 8x + 10y - (.001)(x^2 + xy + y^2)$$

Find the maximum profit.

$P_x = 8 - .001(2x + y) = 0$

$8000 = 2x + y$

$\times 2$

$P_y = 10 - .001(x + 2y) = 0$

$10000 = x + 2y$

$-\ 16000 = 4x + 2y$

$-6000 = -3x$

$x = 2000$

$8000 = 2(2000) + y$

$y = 4000$

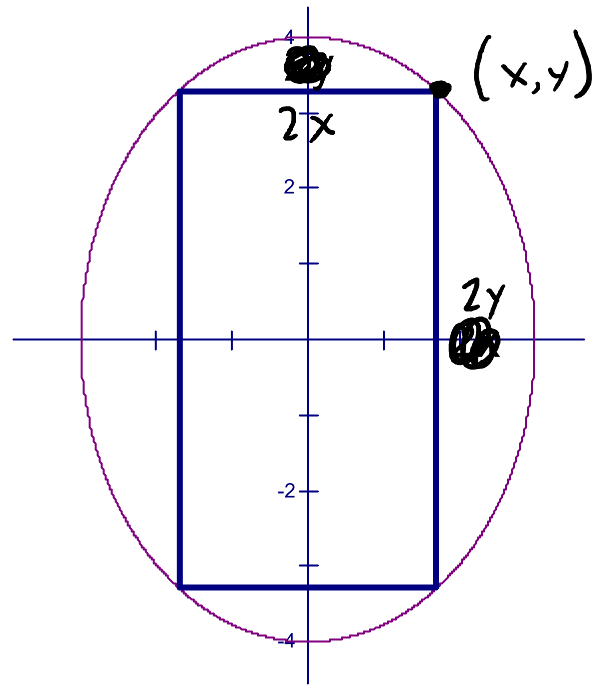
$P(2000, 4000) = 8(2000) + 10(4000) - .001(2000^2 + 2000 \cdot 4000 + 4000^2)$

$= ?$

LaGrange Multipliers

Consider an example from Calculus I:

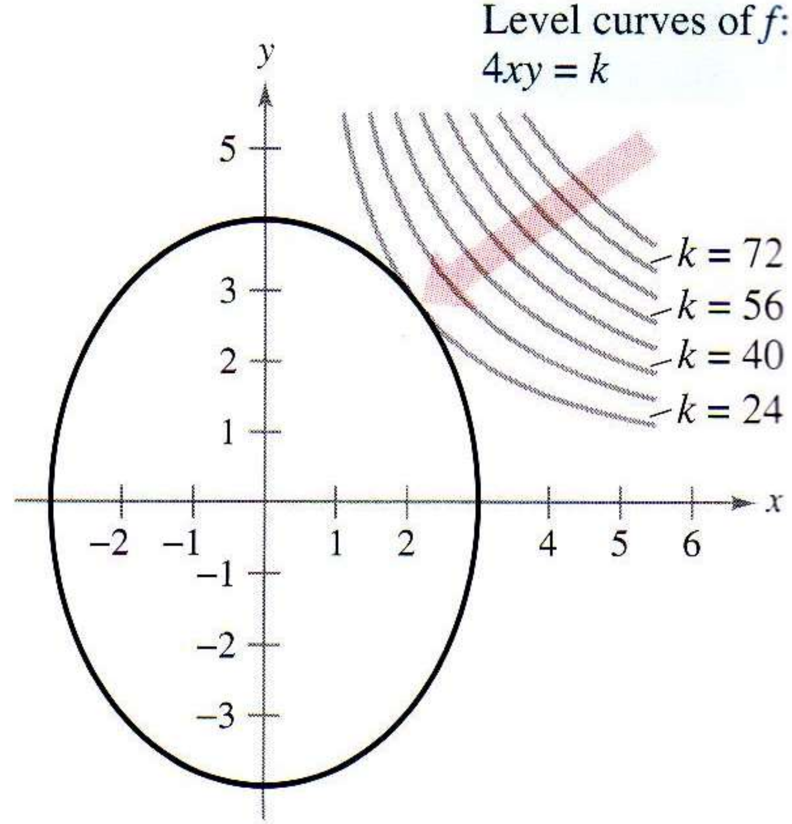
- Find the maximal area of the rectangle bounded by the curve $\frac{x^2}{9}+\frac{y^2}{16}=1$



- Our function $f(x,y)=4xy$ is subject to the constrain $\frac{x^2}{9}+\frac{y^2}{16}=1$

- If we let $g(x,y)=\frac{x^2}{9}+\frac{y^2}{16}$, our constraint is the level curve where $g(x,y) = 1$.

- The family of level curves of $f(x,y)$ is a bunch of hyperbolas. The one that is tangent to the level curve $g(x,y) = 1$ is the maximal one.



Constraint: $g(x, y) = \frac{x^2}{3^2} + \frac{y^2}{4^2} = 1$

- At the point of tangency, both gradients are parallel (remember, gradients are orthogonal to level curves).
- So $\nabla f(x_0,y_0) = \lambda \nabla g(x_0,y_0)$

→ λ is called a <u>LaGrange Multiplier</u>

To find the extreme value of $f(x,y)$ subject to the constraint $g(x,y) = c$:

1. Solve the simultaneous equations $\nabla f = \lambda \nabla g$ and $g(x,y) = c$. Specifically, solve

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x,y) = c \end{cases}$$

2. Evaluate $f(x,y)$ at each solution. The largest is your maximum, the smallest is your minimum.

Ex. Find the extreme values of $f(x,y) = x^2 + 2y^2$ subject to the constraint $\underbrace{x^2 + y^2}_{g} = 1$.

$f_x = \lambda g_x \longrightarrow 2x = \lambda \cdot 2x$

$f_y = \lambda g_y \longrightarrow 4y = \lambda \cdot 2y$

$g = 1 \longrightarrow x^2 + y^2 = 1$

$2x = \lambda \cdot 2x \to x = 0 \to y = \pm 1 \quad (0,1)\ (0,-1)$

$2x = \lambda \cdot 2x \to \lambda = 1 \to 4y = 2y, \quad y = 0, \quad x = \pm 1 \quad (1,0)\ (-1,0)$

$4y = 2\lambda y \longrightarrow \lambda = 2$

$y = 0$

$2x = 4x$

$x = 0$

$2x = 2\lambda x$

$0 = 2\lambda x - 2x$

$0 = 2x(\lambda - 1)$

$2x = 0 \qquad \lambda - 1 = 0$

$f(0,1) = 2$

$f(0,-1) = 2$

$f(1,0) = 1$

$f(-1,0) = 1$

max. value $= 2$

min. value $= 1$

$\underline{\text{Ex.}}$ Find the points on the sphere $\underbrace{x^2+y^2+z^2}_{g}=4$ that are closest to the point (3,1,-1).

$$d=\sqrt{(x-3)^2+(y-1)^2+(z+1)^2} \longrightarrow f(x,y,z)=(x-3)^2+(y-1)^2+(z+1)^2$$

$$f_x=\lambda g_x \longrightarrow \cancel{2}(x-3)=\lambda\cdot\cancel{2}x \longrightarrow \begin{array}{l}-3=x\lambda-x\\ -3=x(\lambda-1)\end{array} \longrightarrow x=\frac{-3}{\lambda-1}$$

$$f_y=\lambda g_y \longrightarrow \cancel{2}(y-1)=\lambda\cdot\cancel{2}y \longrightarrow y=\frac{-1}{\lambda-1}$$

$$f_z=\lambda g_z \longrightarrow \cancel{2}(z+1)=\lambda\cdot\cancel{2}z \longrightarrow z=\frac{1}{\lambda-1}$$

$$g=4 \longrightarrow x^2+y^2+z^2=4$$

$$\left(\frac{-3}{\lambda-1}\right)^2+\left(\frac{-1}{\lambda-1}\right)^2+\left(\frac{1}{\lambda-1}\right)^2=4$$

$$\frac{11}{(\lambda-1)^2}=4$$

$$(\lambda-1)^2=\frac{11}{4}$$

$$\lambda-1=\pm\frac{\sqrt{11}}{2}$$

$$\lambda=1\pm\frac{\sqrt{11}}{2}$$

$$x=\frac{-3}{\pm\frac{\sqrt{11}}{2}}=\mp\frac{6}{\sqrt{11}}$$

$$y=\frac{-1}{\pm\frac{\sqrt{11}}{2}}=\mp\frac{2}{\sqrt{11}}$$

$$z=\frac{1}{\pm\frac{\sqrt{11}}{2}}=\pm\frac{2}{\sqrt{11}}$$

$$\boxed{\left(\frac{-6}{\sqrt{11}},\frac{-2}{\sqrt{11}},\frac{2}{\sqrt{11}}\right)\ \left(\frac{6}{\sqrt{11}},\frac{2}{\sqrt{11}},\frac{-2}{\sqrt{11}}\right)}$$

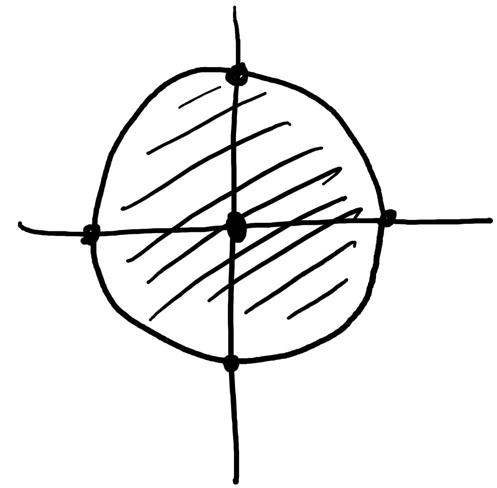
Remember that f is the function to be maximized and $g = c$ is the constraint equation.

This method gives us points on the boundary of a region, but we should still use relative max./min. to check the interior of a region.

Ex. Find the extreme values of $f(x,y) = x^2 + 2y^2$ subject to the constraint $x^2 + y^2 \leq 1$.

Previous problem

$(0,1)\ (0,-1)\ (1,0)\ (-1,0)$



Interior $\rightarrow$ crit. pts.

$f_x = 2x = 0$ $\qquad$ $f_y = 4y = 0$

$x = 0$ $\qquad$ $y = 0$

$(0,0)$

$f(0,1) = 2$

$f(0,-1) = 2$

$f(1,0) = 1$

$f(-1,0) = 1$

$f(0,0) = 0$

max. value $= 2$

min. value $= 0$

If there are two constrains $g(x,y,z) = k$ and $h(x,y,z) = c$, we solve the simultaneous equations:

$$\begin{aligned}
\nabla f &= \lambda \nabla g + \mu \nabla h \\
g(x,y,z) &= k \\
h(x,y,z) &= c
\end{aligned}$$

Ex. Find the maximum value of $f(x,y,z) = x + 2y + 3z$ subject to the constraints $x - y + z = 1$ and $x^2 + y^2 = 1$.