# Tangents and Normals

plane

We're going to talk about tangents and normals to 3-D surfaces such as

line

$x^2 + y^2 + z^2 = 4$

It's useful to think of these surfaces as level curves to functions of 3 variables

→ If $F(x,y,z) = x^2 + y^2 + z^2$, then the surface is the level curve $F(x,y,z) = 4$.

But isn't $\nabla F(x_0, y_0, z_0)$ normal to the level curve containing the points $(x_0, y_0, z_0)$?

<u>Thm.</u> The tangent plane to the level curve $F(x,y,z) = k$ at the point $(x_0, y_0, z_0)$ is

$$\underbrace{F_x(x_0, y_0, z_0)}(x - x_0) + \underbrace{F_y(x_0, y_0, z_0)}(y - y_0) + \underbrace{F_z(x_0, y_0, z_0)}(z - z_0) = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$\vec{n} = \langle a, b, c \rangle$$

Ex. Find the equation of the tangent plane and normal line to $\underbrace{\frac{x^2}{4}+y^2+\frac{z^2}{9}}_{F}=3$ at the point (-2,1,-3).

$$\nabla F = \langle \tfrac{1}{2}x, 2y, \tfrac{2}{9}z \rangle$$

$$\nabla F(-2,1,-3) = \langle -1, 2, -\tfrac{2}{3} \rangle$$

$$\boxed{-1(x+2) + 2(y-1) - \tfrac{2}{3}(z+3) = 0}$$

$$\boxed{\begin{cases} x = -2 - 1t \\ y = 1 + 2t \\ z = -3 - \tfrac{2}{3}t \end{cases}}$$

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases} \qquad \langle a, b, c \rangle$$

What if the surface is defined by $z = f(x,y)$?

If we define $F(x,y,z) = f(x,y) - z$, then we get the equation for our plane:

$$\underbrace{f_x(x_0,y_0)}(x - x_0) + \underbrace{f_y(x_0,y_0)}(y - y_0) \underbrace{-} (z - z_0) = 0$$

$$\nabla F = \langle f_x, f_y, -1 \rangle$$

Ex. Find the equation of the line tangent to the curve of intersection of $z = x^2 + y^2$ and $z = 4 - y$ at (2,-1,5). Are the surfaces orthogonal at that point?

$$0 = \underbrace{x^2+y^2-z}_{F} \qquad 0 = 4 - y - z$$

$$\vec{n} = \langle 0,-1,-1 \rangle$$

$$\nabla F = \langle 2x, 2y, -1 \rangle$$

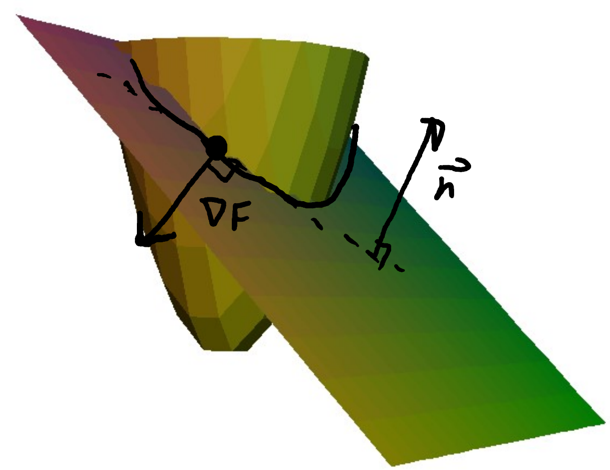
$$\nabla F(2,-1,5) = \langle 4,-2,-1 \rangle$$

$$\vec{v} = \nabla F \times \vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -1 & -1 \\ 4 & -2 & -1 \end{vmatrix} = \langle -1, -4, 4 \rangle$$

$$\boxed{\begin{cases} x = 2 - 1t \\ y = -1 - 4t \\ z = 5 + 4t \end{cases}}$$

$$\nabla F \cdot \vec{n} = 0 + 2 + 1 = 3 \neq 0$$

not orthog.



Ex. Find the angle of inclination of the tangent plane to $\underbrace{\frac{x^2}{12}+\frac{y^2}{12}+\frac{z^2}{3}}_{F}=1$ at (2,2,1).

[The angle between the tangent plane and the *xy*-plane.]

$z=0$

$\vec{n}=\nabla F(2,2,1)=\langle \frac{1}{3}, \frac{1}{3}, \frac{2}{3}\rangle$

$\nabla F=\langle \frac{1}{6}x, \frac{1}{6}y, \frac{2}{3}z\rangle$

$\hat{k}=\langle 0,0,1\rangle$

$\vec{n}\cdot\hat{k}=\|\vec{n}\|\,\|\hat{k}\|\cos\theta$

$\frac{2}{3}=\sqrt{\frac{1}{9}+\frac{1}{9}+\frac{4}{9}}\cdot 1\cdot\cos\theta$

$\frac{2}{3}=\frac{\sqrt{6}}{3}\cos\theta$

$\cos\theta=\frac{2}{\sqrt{6}}$

# Extreme Values

Let $f(x,y)$ be defined on a region $R$ containing $P(x_0,y_0)$:

- $P$ is a <u>relative max</u> of $f$ if $f(x,y) \leq f(x_0,y_0)$ for all $(x,y)$ on an open disk containing $P$.
- $P$ is a <u>relative min</u> of $f$ if $f(x,y) \geq f(x_0,y_0)$ for all $(x,y)$ on an open disk containing $P$.

$(x_0,y_0)$ is a critical point of $f$ if either

- $\nabla f(x_0,y_0) = \mathbf{0}$ or

  $f_x = 0$ and $f_y = 0$

- $f_x(x_0,y_0)$ or $f_y(x_0,y_0)$ is undefined.

Thm. If point $P$ is a relative extrema, then it is a critical point.

Ex. Find and classify the relative extrema of

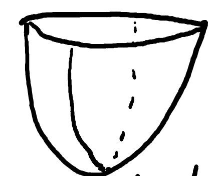
$$f(x,y) = x^2 + y^2 - 2x - 6y + 14$$

$f_x = 2x - 2 = 0$

$x = 1$

$f_y = 2y - 6 = 0$

$y = 3$



paraboloid

$(1,3)$ crit. pt.

rel. min.

**Ex.** Find and classify the relative extrema of

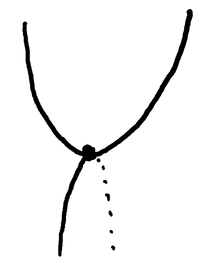
$$f(x,y) = y^2 - x^2$$

$f_x = -2x = 0$ $\qquad$ $f_y = 2y = 0$

$x = 0$ $\qquad$ $y = 0$

$(0,0)$ crit. pt.

neither max. nor min.

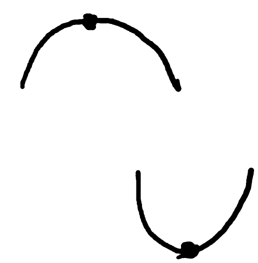


hyperbolic paraboloid

An easier way to classify critical points is the Second Derivatives Test.

<u>Thm.</u> Second Partial Derivatives Test

Let $f(x,y)$ have continuous second partial derivatives on an open region containing $(a,b)$ such that $\nabla f(a,b) = \mathbf{0}$. Define



$$d = f_{xx}(a,b)\, f_{yy}(a,b) - [\, f_{xy}(a,b)]^2$$

1) If $d > 0$ and $f_{xx}(a,b) < 0$, then $(a,b)$ is a rel. max.
2) If $d > 0$ and $f_{xx}(a,b) > 0$, then $(a,b)$ is a rel. min.
3) If $d < 0$, then $(a,b)$ is a saddle point.
4) If $d = 0$, then the test fails.

Ex. Find and classify the relative extrema of

$$f(x,y)=x^4+y^4-4xy+1$$

$f_x = 4x^3 - 4y = 0$

$y = x^3$

$y = (y^3)^3$

$y = y^9$

$0 = y^9 - y$

$0 = y(y^8 - 1)$

$y = 0, 1, -1$

$f_y = 4y^3 - 4x = 0$

$x = y^3$

$y=0,\ x=0$ $(0,0)$

$y=1,\ x=1$ $(1,1)$

$y=-1,\ x=-1$ $(-1,-1)$

$f_{xx} = 12x^2$ $f_{yy} = 12y^2$

$f_{xy} = -4$

| | $f_{xx}$ | $f_{yy}$ | $f_{xy}$ | d | classify |
|---|---|---|---|---|---|
| (0,0) | 0 | 0 | -4 | -16 | saddle |
| (1,1) | 12 | 12 | -4 | 128 | rel. min. |
| (-1,-1) | 12 | 12 | -4 | 128 | rel. min. |

Ex. Find and classify the relative extrema of

$$f(x,y) = -x^3 - 3x + 4x^2y - 2xy^2 + 1$$

$f_x = -3x^2 - 3 + 8xy - 2y^2 = 0$

$f_y = 4x^2 - 4xy = 0$

$4x(x-y) = 0$

$x = 0 \qquad x = y$

Case 1: $x=0$

$-3 - 2y^2 = 0$

$\cancel{y^2 = -\frac{3}{2}}$

Case 2: $x=y$

$-3x^2 - 3 + 8x^2 - 2x^2 = 0$

$3x^2 - 3 = 0$

$x^2 = 1$

$x = \pm 1$

$(1,1) \quad (-1,-1)$

$f_{xx} = -6x + 8y$

$f_{yy} = -4x$

$f_{xy} = 8x - 4y$

| | $f_{xx}$ | $f_{yy}$ | $f_{xy}$ | d | classify |
|---|---|---|---|---|---|
| $(1,1)$ | 2 | -4 | 4 | -24 | saddle |
| $(-1,-1)$ | -2 | 4 | -4 | -24 | saddle |

Ex. Find the shortest distance from the point (1,0,-2) to the plane $x + 2y + z = 4$.

Point on plane $(x, y, z)$

$z = 4 - x - 2y$

$$d = \sqrt{(x-1)^2 + y^2 + (z+2)^2}$$

$$D = (x-1)^2 + y^2 + (4-x-2y+2)^2$$

$$D = (x-1)^2 + y^2 + (6-x-2y)^2$$

$$D_x = 2(x-1) + 2(6-x-2y)(-1) = 0$$

$$2x - 2 - 12 + 2x + 4y = 0$$

$$4x + 4y = 14$$

$$2x + 2y = 7$$

$$D_y = 2y + 2(6-x-2y)(-2) = 0$$

$$2y - 24 + 4x + 8y = 0$$

$$4x + 10y = 24$$

$$2x + 5y = 12$$

$$-\ 2x + 2y = 7$$

$$3y = 5$$

$$y = \frac{5}{3}$$

$$2x + 2\left(\frac{5}{3}\right) = 7$$

$$2x = 7 - \frac{10}{3}$$

$$x = \frac{11}{6}$$

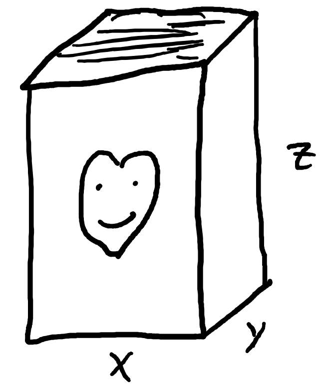
$$z = 4 - \frac{11}{6} - 2\left(\frac{5}{3}\right)$$

$$z = \frac{-7}{6}$$

$$\left(\frac{11}{6}, \frac{5}{3}, \frac{-7}{6}\right)$$

$$d = \sqrt{\left(\frac{11}{6} - 1\right)^2 + \left(\frac{5}{3}\right)^2 + \left(\frac{-7}{6} + 2\right)^2} = ?$$

Ex. A rectangular box without a lid is made from 12 m$^2$ of cardboard. Find the maximum volume.




$xy + 2xz + 2yz = 12 \longrightarrow z(2x+2y) = 12 - xy$

$$z = \frac{12 - xy}{2x + 2y}$$

$$V = xyz = xy\,\frac{12 - xy}{2x + 2y}$$

$$V = \frac{12xy - x^2y^2}{2x + 2y}$$

$$V_x = \frac{(2x+2y)(12y - 2xy^2) - (12xy - x^2y^2)\cdot 2}{(2x+2y)} = \frac{\cancel{24xy} - 4x^2y^2 + 24y^2 - 4xy^3 - \cancel{24xy} + 2x^2y^2}{(2x+2y)^2}$$

$$= \frac{-2x^2y^2 + 24y^2 - 4xy^3}{(2x+2y)^2} \longrightarrow -2x^2y^2 + 24y^2 - 4xy^3 = 0 \rightarrow 2y^2(-x^2 + 12 - 2xy) = 0$$

$$-x^2 + 12 - 2xy = 0$$

$V_y = \longrightarrow$

$$-y^2 + 12 - 2xy = 0$$

$$-x^2 + y^2 = 0$$

$$y = \pm x$$

Case 1: $y = x$ $\quad -x^2 + 12 - 2x^2 = 0$

$12 = 3x^2$

$x^2 = 4$

$x = \pm 2$

$(2, 2)$

$\cancel{(-2, -2)}$

Case 2: $y = -x$ $\quad -x^2 + 12 + 2x^2 = 0$

$x^2 = -12$ X

$$V = \frac{12(2)(2) - 2^2 2^2}{2\cdot 2 + 2\cdot 2} = \frac{32}{8} = 4$$

To find the absolute max/min values of $f$ on a closed region $D$:

1) Find the value of $f$ at any critical point that lie in $D$.

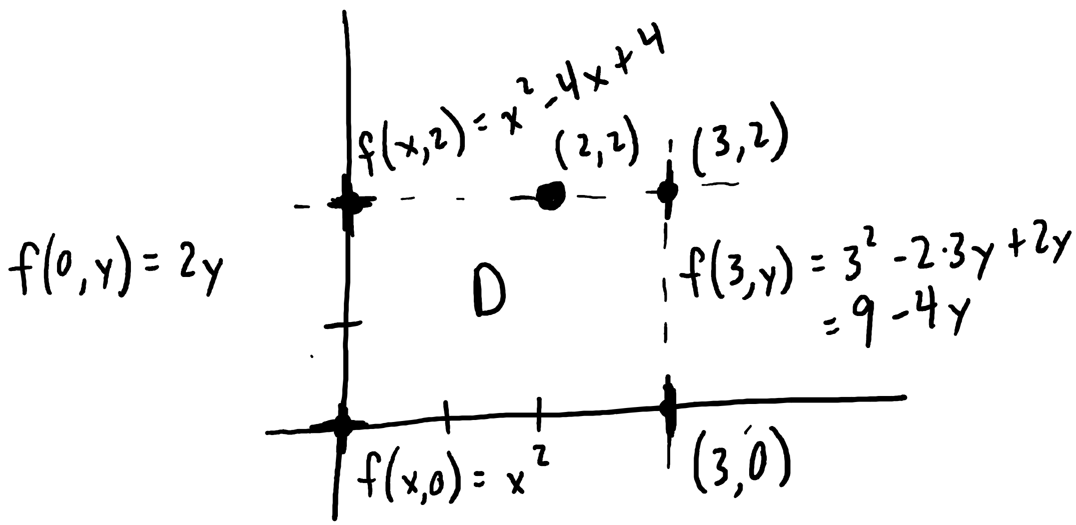
2) Find the extreme values of $f$ on the boundary of $D$.

→ The largest value is the absolute max., the smallest value is the absolute min.

Ex. Find the extreme values of $f(x,y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x,y) \mid 0 \le x \le 3,\ 0 \le y \le 2\}$.

Crit. pts $\quad f_x = 2x - 2y = 0 \qquad f_y = -2x + 2 = 0 \qquad (1, 1)$

$\qquad\qquad y = x \qquad\qquad\qquad x = 1$




$f(1,1) = 1 - 2 + 2 = 1$

$f(3,2) = 9 - 12 + 4 = 1$

$f(3,0) = 9 + 0 + 0 = 9$

$f(0,0) = 0 + 0 + 0 = 0$

$f(0,2) = 0 - 0 + 4 = 4$

$f(2,2) = 4 - 8 + 4 = 0$

abs. max. value is 9

abs. min. value is 0

Ex. Find the extreme values of $f(x,y) = 1 + 4x - 5y$ on the triangular region $D$ with vertices $(0,0)$, $(2,0)$, and $(0,3)$.

$f_x = 4 \neq 0 \qquad f_y = -5 \neq 0$ no crit. pts.

$f(0,0) = 1 + 0 + 0 = 1$

$f(0,3) = 1 + 0 - 15 = -14$ min.

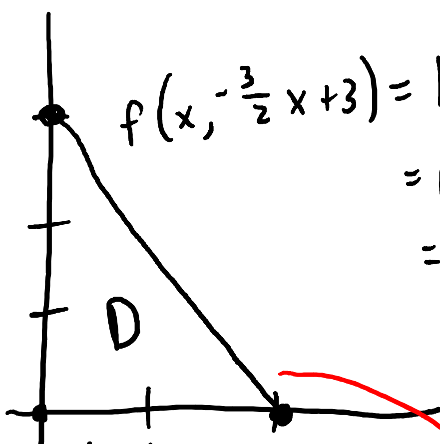
$f(2,0) = 1 + 8 + 0 = 9$ max.

$f\left(x, -\frac{3}{2}x + 3\right) = 1 + 4x - 5\left(\frac{-3}{2}x + 3\right)$

$= 1 + 4x + \frac{15}{2}x - 15$

$= \frac{23}{2}x - 14$

$f(0,y) = 1 - 5y$




$f(x,0) = 1 + 4x$

$y = \frac{-3}{2}x + 3$