Related Rates

p. 279: 1-11, 13-19

1. $V = x^3 \Rightarrow \frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt} = 3x^2\frac{dx}{dt}$

2. (a) $A = \pi r^2 \Rightarrow \frac{dA}{dt} = \frac{dA}{dr}\frac{dr}{dt} = 2\pi r\frac{dr}{dt}$ (b) $\frac{dA}{dt} = 2\pi r\frac{dr}{dt} = 2\pi(30\text{ m})(1\text{ m/s}) = 60\pi\text{ m}^2/\text{s}$

3. Let s denote the side of a square. The square's area A is given by $A = s^2$. Differentiating with respect to t gives us $\frac{dA}{dt} = 2s\frac{ds}{dt}$. When A = 16, s = 4. Substituting 4 for s and 6 for $\frac{ds}{dt}$ gives us $\frac{dA}{dt} = 2(4)(6) = 48\text{ cm}^2/\text{s}$.

4. $A = \ell w \Rightarrow \frac{dA}{dt} = \ell\cdot\frac{dw}{dt} + w\cdot\frac{d\ell}{dt} = 20(3) + 10(8) = 140\text{ cm}^2/\text{s}$.

5. $V = \pi r^2 h = \pi(5)^2 h = 25\pi h \Rightarrow \frac{dV}{dt} = 25\pi\frac{dh}{dt} \Rightarrow 3 = 25\pi\frac{dh}{dt} \Rightarrow \frac{3}{25\pi}$ m/min.

6. $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dS}{dt} = \frac{4}{3}\pi\cdot 3r^2\frac{dr}{dt} \Rightarrow \frac{dV}{dt} = 4\pi\left(\frac{1}{2}\cdot 80^2\right)(4) = 25{,}600\pi\text{ mm}^3/\text{s}$.

7. $S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = 4\pi\cdot 2r\frac{dr}{dt} \Rightarrow \frac{dS}{dt} = 4\pi\cdot 2\cdot 8\cdot 2 = 128\pi\text{ cm}^2/\text{min}$.

8. (a) $A = \frac{1}{2}ab\sin\theta \Rightarrow \frac{dA}{dt} = \frac{1}{2}ab\cos\theta\frac{d\theta}{dt} = \frac{1}{2}(2)(3)(\cos\frac{\pi}{3})(0.2) = 3(\frac{1}{2})(0.2) = 0.3\text{cm}^2/\text{min}$.

   (b) $A = \frac{1}{2}ab\sin\theta \Rightarrow \frac{dA}{dt} = \frac{1}{2}a\left(b\cos\theta\frac{d\theta}{dt} + \sin\theta\frac{db}{dt}\right) = \frac{1}{2}(2)[3(\cos\frac{\pi}{3})(0.2) + 3(\sin\frac{\pi}{3})(1.5)]$

   $= 3\left(\frac{1}{2}\right)(0.2) + \frac{1}{2}\sqrt{3}\left(\frac{3}{2}\right) = 0.3 + \frac{3}{4}\sqrt{3} \approx 1.6\text{ cm}^2/\text{min}$.

   (c) $A = \frac{1}{2}ab\sin\theta \Rightarrow \frac{dA}{dt} = \frac{1}{2}\left(\frac{da}{dt}b\sin\theta + a\frac{db}{dt}\sin\theta + ab\cos\theta\frac{d\theta}{dt}\right)$

   $= \frac{1}{2}[2.5(3)\left(\frac{1}{2}\sqrt{3}\right) + 2(1.5)\left(\frac{1}{2}\sqrt{3}\right) + 2(3)\left(\frac{1}{2}\right)(0.2)]$

   $= \frac{15}{8}\sqrt{3} + \frac{3}{4}\sqrt{3} + 0.3 = \frac{21}{8}\sqrt{3} + 0.3 \approx 4.85\text{ cm}^2/\text{min}$.

   Note how this answer relates the answer in part (a) [θ changing] and part (b) [b and θ changing].

9. (a) $y = \sqrt{2x+1}$ and $\frac{dx}{dt} = 3 \Rightarrow \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{1}{2}(2x+1)^{-1/2}\cdot 2\cdot 3 = \frac{3}{\sqrt{2x+1}}$. When $x = 4, \frac{dy}{dt} = \frac{3}{\sqrt{9}} = 1$.

   (b) $y = \sqrt{2x+1} \Rightarrow y^2 = 2x+1 \Rightarrow 2x = y^2 - 1 \Rightarrow x = \frac{1}{2}y^2 - \frac{1}{2}$ and $\frac{dy}{dt} = 5 \Rightarrow$

   $\frac{dx}{dt} = \frac{dx}{dy}\frac{dy}{dt} = y\cdot 5 = 5y$. When $x = 12, y = \sqrt{25} = 5$, so $\frac{dx}{dt} = 5(5) = 25$.

10. (a) $\frac{d}{dt}(4x^2 + 9y^2) = \frac{d}{dt}(36) \Rightarrow 8x\frac{dx}{dt} + 18y\frac{dy}{dt} = 0 \Rightarrow 4x\frac{dx}{dt} + 9y\frac{dy}{dt} = 0 \Rightarrow$

    $4(2)\frac{dx}{dt} + 9\left(\frac{2}{3}\sqrt{5}\right)\left(\frac{1}{3}\right) = 0 \Rightarrow 8\frac{dx}{dt} = -2\sqrt{5} \Rightarrow \frac{dx}{dt} = -\frac{1}{4}\sqrt{5}$

    (b) $4x\frac{dx}{dt} + 9y\frac{dy}{dt} = 0 \Rightarrow 4(-2)(3) + 9\left(\frac{2}{3}\sqrt{5}\right)\frac{dy}{dt} = 0 \Rightarrow 6\sqrt{5}\frac{dy}{dt} = 24 \Rightarrow \frac{dy}{dt} = \frac{4}{\sqrt{5}}$

11. $xy = 20 \Rightarrow x\frac{dy}{dt} + y\frac{dx}{dt} = 0 \Rightarrow \frac{dx}{dt} = -\frac{x}{y}\frac{dy}{dt}$. When $x = 4$, $y = \frac{20}{4} = 5$, and when $\frac{dy}{dx} = -2$,

$\frac{dx}{dt} = -\frac{4}{5}(-2) = 1.6$ **(C)**

13. $\frac{d}{dt}(x^2 + y^2 + z^2) = \frac{d}{dt}(9) \Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2z\frac{dz}{dt} = 0 \Rightarrow x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt} = 0.$

If $\frac{dx}{dt} = 5$, $\frac{dy}{dt} = 4$ and $(x, y, z) = (2, 2, 1)$, then $2(5) + 2(4) + 1\frac{dz}{dt} = 0 \Rightarrow \frac{dz}{dt} = -18.$

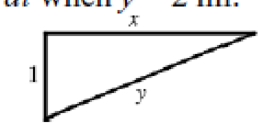
14. $\frac{d}{dt}(xy) = \frac{d}{dt}(8) \Rightarrow x\frac{dy}{dt} + y\frac{dy}{dt} = 0$. If $\frac{dy}{dt} = -3$ cm/s and $(x, y) = (4, 2)$, then $4(-3) + 2\frac{dx}{dt} = 0 \Rightarrow$

$\frac{dx}{dt} = 6$. Thus, the x-coordinate is increasing at a rate of 6 cm/s.

15. (a) Given: a plane flying horizontally at an altitude of 1 mi and a speed of 500 mi/h passes directly over a radar station. If we let t be time (in hours) and x be the horizontal distance traveled by the plane (in miles), then we are given that $dx/dt = 500$ mi/h.

(b) Unknown: the rate at which the distance from the plane to the station is increasing when it is 2 mi from the station. If we let y be the distance from the plane to the station, then we want to find dy/dt when $y = 2$ mi.

(c)



(d) By the Pythagorean Theorem, $y^2 = x^2 + 1 \Rightarrow 2y(dy/dt) = 2x(dx/dt)$.

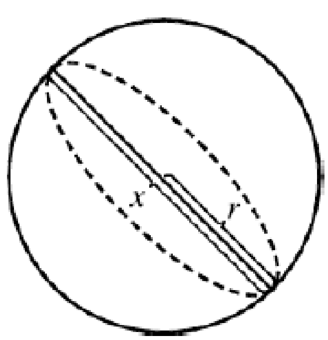
(e) $\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt} = \frac{x}{y}(500)$. Since $y^2 = x^2 + 1$, when $y = 2, x = \sqrt{3}$, so

$\frac{dy}{dt} = \frac{\sqrt{3}}{2}(500) = 250\sqrt{3} \approx 433$ mi/h.

16. (a) Given: the rate of decrease of the surface are is 1 cm^2/min. If we let t be time (in minutes) and S be the surface area (in cm^2), then we are given that $dS/dt = -1$ cm^2/min.

(b) Unknown: the rate of decrease of the diameter is 10 cm. If we let x be the diameter, then we want to find dx/dt when $x = 10$ cm.

(c)



(d) If the radius is r and the diameter $x = 2r$, then $r = \frac{1}{2}x$ and

$S = 4\pi r^2 = 4\pi\left(\frac{1}{2}x\right)^2 = \pi x^2 \Rightarrow \frac{dS}{dt} = \frac{dS}{dx}\frac{dx}{dt} = 2\pi x\frac{dx}{dt}.$

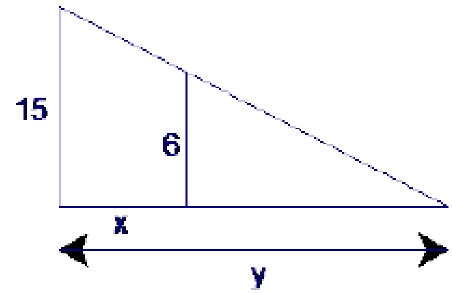
(e) $-1 = \frac{dS}{dt} = 2\pi x\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = -\frac{1}{2\pi x}.$

When $x = 10$, $\frac{dx}{dt} = -\frac{1}{20\pi}$. So the rate of decrease is $\frac{1}{20\pi}$ cm/min.

17.

$$x = 40, \frac{dx}{dt} = 5, \frac{dy}{dt} = ?$$

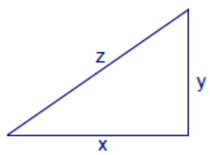
$$\frac{y}{15} = \frac{y-x}{6}$$
$$6y = 15y - 15x$$
$$15x = 9y$$
$$y = \frac{5}{3}x$$
$$\frac{dy}{dt} = \frac{5}{3}\frac{dx}{dt} = \frac{5}{3}(5) = \frac{25}{3}$$



So the tip of the shadow is moving at a rate of $\frac{25}{3}$ ft/s.

18.

$$\frac{dx}{dt} = -35, \frac{dy}{dt} = 25$$
$$x = 150 - 35(4) = 10$$
$$y = 25(4) = 100$$
$$\frac{dz}{dt} = ?$$



$$z^2 = x^2 + y^2$$
$$z = \sqrt{10^2 + 100^2} = 100.499$$

$$z^2 = x^2 + y^2$$
$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$
$$2(100.499)\frac{dz}{dt} = 2(10)(-35) + 2(100)(25)$$
$$\frac{dz}{dt} = 21.393 \text{ km/h}$$

19. We are given that $\frac{dx}{dt} = 60$ mi/h and $\frac{dy}{dt} = 25$ mi/h. $z^2 = x^2 + y^2 \Rightarrow$

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \Rightarrow z\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} \Rightarrow \frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right).$$

After 2 hours, $x = 2(60) = 120$ and $y = 2(25) = 50 \Rightarrow \quad z = \sqrt{120^2 + 50^2} = 130$, so

$$\frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) = \frac{120(60) + 50(25)}{130} = 65 \text{ mi/h}.$$