L'Hopital's Rule

p. 347: 5, 12, 13-39 odd (skip 33), 43, 73, 81

5. (a) $\lim_{x\to a}\frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$.

(b) $\lim_{x\to a}\frac{f(x)}{p(x)}=0$ because the numerator approaches 0 while the denominator becomes large.

(c) $\lim_{x\to a}\frac{h(x)}{p(x)}=0$ because the numerator approaches a finite number while the denominator becomes large.

(d) If $\lim_{x\to a}p(x)=\infty$ and $f(x)\to 0$ through positive values, then $\lim_{x\to a}\frac{p(x)}{f(x)}=\infty$. $\left[\text{For example, take } a=0,\ p(x)=1/x^2, \text{ and } f(x)=x^2.\right]$ If $f(x)\to 0$ through negative values, then $\lim_{x\to a}\frac{p(x)}{f(x)}=-\infty$. $\left[\text{For example, } a=0,\ p(x)=1/x^2, \text{ and } f(x)=-x^2.\right]$ If $f(x)\to 0$ through both positive and negative value, then the limit might not exist. $\left[\text{For example, } a=0,\ p(x)=1/x^2, \text{ and } f(x)=x.\right]$

(e) $\lim_{x\to a}\frac{p(x)}{q(x)}$ is an indeterminate form of type $\frac{\infty}{\infty}$.

12. L'Hopital's Rule does not apply to the expression (C) $\lim_{x\to -2}\frac{x^2-4}{x-2}$ because this limit is not of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ $\left(\lim_{x\to -1}x-2=-4\neq 0.\right)$

13. This limit has the form $\frac{0}{0}$. $\lim_{x\to 3}\frac{x-3}{x^2-9}=\lim_{x\to 3}\frac{x-3}{(x-3)(x+3)}=\lim_{x\to 3}\frac{1}{x+3}=\frac{1}{3+3}=\frac{1}{6}$.

Note: Alternatively, we could apply l'Hopital's Rule.

15. This limit has the form $\frac{0}{0}$. $\lim_{x\to -2}\frac{x^3+8}{x+2}\overset{\text{H}}{=}\lim_{x\to -2}\frac{3x^2}{1}=3(-2)^2=12$.

Note: Alternatively, we could factor and simplify.

17. This limit has the form $\frac{0}{0}$. $\lim_{x\to 1/2}\frac{6x^2+5x-4}{4x^2+16x-9}\overset{\text{H}}{=}\lim_{x\to 1/2}\frac{12x+5}{8x+16}=\frac{6+5}{4+16}=\frac{11}{20}$.

Note: Alternatively, we could factor and simplify.

19. This limit has the form $\frac{0}{0}$. $\lim_{x\to 0}\frac{\tan 3x}{\sin 2x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{3\sec^2 x}{2\cos 2x}=\frac{3(1^2)}{2(1)}=\frac{3}{2}$.

21. This limit has the form $\frac{0}{0}$. $\lim_{x\to 0}\frac{x^2}{1-\cos x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{2x}{\sin x}=\lim_{x\to 0}\frac{2}{(\sin x)/x}=\frac{2}{1}=2$.

23. This limit can be evaluated by substituting π for θ. $\lim_{\theta\to\pi}\frac{1+\cos\theta}{1-\cos\theta}=\frac{1+(-1)}{1-(-1)}=\frac{0}{2}=0$.

25. This limit has the form $\frac{\infty}{\infty}$. $\lim_{x\to\infty}\frac{x+x^2}{1-2x^2} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{1+2x}{-4x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{2}{-4}=-\frac{1}{2}$.

A better method is to divide the numerator and denominator by x^2:

$$\lim_{x\to\infty}\frac{x+x^2}{1-2x^2}=\lim_{x\to\infty}\frac{\frac{1}{x}+1}{\frac{1}{x^2}-2}=\frac{0+1}{0-2}=-\frac{1}{2}.$$

27. This limit has the form $\frac{\infty}{\infty}$. $\lim_{x\to\infty}\frac{\ln\sqrt{x}}{x^2}=\lim_{x\to\infty}\frac{\frac{1}{2}\ln x}{x^2} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{\frac{1}{2x}}{2x}=\lim_{x\to\infty}\frac{1}{4x^2}=0.$

29. This limit has the form $\frac{0}{0}$. $\lim_{t\to0}\frac{8^t-5^t}{t} \overset{\text{H}}{=} \lim_{t\to0}\frac{8^t\ln 8-5^t\ln 5}{1}=\ln 8-\ln 5=\ln\frac{8}{5}.$

31. This limit has the form $\frac{\infty}{\infty}$.

$$\lim_{u\to\infty}\frac{e^{u/10}}{u^3} \overset{\text{H}}{=} \lim_{u\to\infty}\frac{e^{u/10}\cdot\frac{1}{10}}{3u^2} \overset{\text{H}}{=} \frac{1}{30}\lim_{u\to\infty}\frac{e^{u/10}\cdot\frac{1}{10}}{2u} \overset{\text{H}}{=} \frac{1}{600}\lim_{u\to\infty}\frac{e^{u/10}\cdot\frac{1}{10}}{1}=\frac{1}{6000}\lim_{u\to\infty}e^{u/10}=\infty$$

35. This limit has the form $\frac{\infty}{\infty}$. $\lim_{x\to\infty}\frac{(\ln x)^2}{x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{2(\ln x)(1/x)}{1}=2\lim_{x\to\infty}\frac{\ln x}{x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{1/x}{1}=2(0)=0$

37. This limit has the form $\frac{0}{0}$.

$$\lim_{x\to0}\frac{\cos mx-\cos nx}{x^2} \overset{\text{H}}{=} \lim_{x\to0}\frac{-m\sin mx+n\sin nx}{2x} \overset{\text{H}}{=} \lim_{x\to0}\frac{-m^2\cos mx+n^2\cos nx}{2}=\tfrac{1}{2}\left(n^2-m^2\right).$$

39. This limit has the form $\frac{0}{0}$. $\lim_{x\to1}\frac{x\sin(x-1)}{2x^2-x-1} \overset{\text{H}}{=} \lim_{x\to1}\frac{x\cos(x-1)+\sin(x-1)}{4x-1}=\frac{\cos 0}{4-1}=\frac{1}{3}.$

43. This limit has the form $\frac{0}{0}$. $\lim_{x\to0}\frac{e^x-e^{-x}-2x}{x-\sin x} \overset{\text{H}}{=} \lim_{x\to0}\frac{e^x+e^{-x}-2}{1-\cos x} \overset{\text{H}}{=} \lim_{x\to0}\frac{e^x-e^{-x}}{\sin x} \overset{\text{H}}{=} \lim_{x\to0}\frac{e^x+e^{-x}}{\cos x}=\frac{1+1}{1}=2.$

73. $\lim_{x\to1}\frac{xe-e^x}{(x+1)^2-1}=\frac{e-e^1}{(1+1)^2-1}=\frac{0}{4-1}=0$, which is choice (**D**).

81. $\lim_{x\to0}\frac{f(x)-x-1}{\sin(2x)-x^2-2x} \overset{\text{H}}{=} \lim_{x\to0}\frac{f'(x)-1}{2\cos(2x)-2x-2} \overset{\text{H}}{=} \lim_{x\to0}\frac{f''(x)}{-2\sin(2x)-2}=\frac{2}{-2}=-1$, which is option (**A**).