p. 671: 25-49 odd, 53, 62-64, 68, 69-75 odd, 78-82

25. (a) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{3n+1} = \frac{2}{3}$, so the *sequence* $\{a_n\}$ is convergent.

(b) Because $\lim_{n\to\infty} a_n = \frac{2}{3} \neq 0$, the *series* $\sum_{n=1}^{\infty} a_n$ is divergent by the Test for Divergence.

27. $3 - 4 + \frac{16}{3} - \frac{64}{9} + \cdots$ is a geometric series with ratio $r = -\frac{4}{3}$. Since $|r| = \frac{4}{3} > 1$, the series diverges.

29. $10 - 2 + 0.4 - 0.08 + \cdots$ is a geometric series with ratio $r = -\frac{2}{10} = -\frac{1}{5}$. Since $|r| = \frac{1}{5} < 1$, the series converges to $\frac{a}{1-r} = \frac{10}{1-(-1/5)} = \frac{10}{6/5} = \frac{50}{6} = \frac{25}{3}$.

31. $\sum_{n=1}^{\infty} 12(0.73)^{n-1}$ is a geometric series with first term $a = 12$ and ratio $r = 0.73$. Since $|r| = 0.73 < 1$, the series converges to $\frac{a}{1-r} = \frac{12}{1-0.73} = \frac{12}{0.27} = \frac{12(100)}{27} = \frac{400}{9}$.

33. $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \frac{1}{4}\sum_{n=1}^{\infty}\left(-\frac{3}{4}\right)^{n-1}$. The latter series is geometric series with $a = 1$ and ratio $r = -\frac{3}{4}$. Since $|r| = \frac{3}{4} < 1$, it converges to $\frac{1}{1-(-3/4)} = \frac{4}{7}$. Thus, the given series converges to $\frac{1}{4}\left(\frac{4}{7}\right) = \frac{1}{7}$.

35. $\sum_{n=1}^{\infty} \frac{e^{2n}}{6^{n-1}} = \sum_{n=1}^{\infty} \frac{(e^2)^n}{6^n 6^{-1}} = 6\sum_{n=1}^{\infty}\left(\frac{e^2}{6}\right)^n$ is a geometric series with ratio $r = \frac{e^2}{6}$. Since $|r| = \frac{e^2}{6} \approx 1.23 > 1$, the series diverges.

37. $3 - \frac{2}{3} + \frac{4}{27} - \frac{8}{243} + \cdots = \sum_{n=0}^{\infty} 3\left(\frac{-2}{9}\right)^n$ is a geometric series with $a = 3$ and $r = -\frac{2}{9}$. Because $|r| = \frac{2}{9} < 1$, the series converges to $\frac{a}{1-r} = \frac{3}{1-(-2/9)} = \frac{3}{11/9} = \frac{27}{11}$, choice (C).

39. $\frac{1}{3} + \frac{2}{9} + \frac{1}{27} + \frac{2}{81} + \frac{1}{243} + \frac{2}{729} + \cdots = \left(\frac{1}{3} + \frac{1}{27} + \frac{1}{243} + \cdots\right) + \left(\frac{2}{9} + \frac{2}{81} + \frac{2}{729} + \cdots\right)$, which are both convergent geometric series with sums $\frac{1/3}{1-1/9} = 3/8$ and $\frac{2/9}{1-1/9} = 1/4$, so the original series converges to $\frac{3}{8} + \frac{1}{4} = \frac{5}{8}$.

41. $\sum_{n=1}^{\infty} \frac{k^2}{k^2 - 2k + 5}$ diverges by the Test for Divergence because
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{k^2}{k^2 - 2k + 5} = \lim_{n\to\infty} \frac{1}{1 - 2/k + 5/k^2} = 1 \neq 0.$$

43.
$$\sum_{n=1}^{\infty}\left[(-0.2)^n + (0.6)^{n-1}\right] = \sum_{n=1}^{\infty}(-0.2)^n + \sum_{n=1}^{\infty}(0.6)^{n-1} \quad \text{[sum of two geometric series]}$$
$$= \frac{-0.2}{1-(-0.2)} + \frac{1}{1-0.6} = -\frac{1}{6} + \frac{5}{2} = \frac{7}{3}$$
The series converges to $\frac{7}{3}$.

45. $\sum_{n=1}^{\infty} \frac{2^n+4^n}{e^n}$ diverges by the Test for Divergence because $\lim_{n\to\infty} \frac{2^n+4^n}{e^n} = \lim_{n\to\infty}\left(\frac{2^n}{e^n}+\frac{4^n}{e^n}\right)$ $\geq \lim_{n\to\infty}\left(\frac{4}{e^n}\right)^n = \infty$ since $\frac{4}{e} > 1$.

47. $\sum_{n=1}^{\infty} \frac{1}{1+\left(\frac{2}{3}\right)^n}$ diverges by the Test for Divergence because $\lim_{n\to\infty} \frac{1}{1+\left(\frac{2}{3}\right)^n} = \frac{1}{1+0} = 1 \neq 0$.

49. $\sum_{k=0}^{\infty}\left(\sqrt{2}\right)^{-k} = \sum_{k=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^k$ is a geometric series with first term $a = \left(\frac{1}{\sqrt{2}}\right)^0 = 1$ and ratio $r = \frac{1}{\sqrt{2}}$. Because $|r| < 1$, the series converges to $\frac{1}{1-1/\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}-1} \approx 3.414$.

53. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^n}{n^2} = \lim_{x\to\infty} \frac{e^x}{x^2} \overset{H}{=} \lim_{x\to\infty} \frac{e^x}{2x} \overset{H}{=} \lim_{x\to\infty} \frac{e^x}{2} = \infty \neq 0$, so $\sum_{n=1}^{\infty} \frac{e^n}{n^2}$ diverges by the Test for Divergence.

62. $0.\overline{8} = \frac{8}{10} + \frac{8}{10^2} + \frac{8}{10^3} + \cdots$ is a geometric series with $a = 8/10$ and $r = 1/10$. It converges to $\frac{8/10}{1-1/10} = \frac{8}{9}$.

63. $0.\overline{46} = \frac{46}{100} + \frac{46}{100^2} + \cdots$ is a geometric series with $a = \frac{46}{100}$ and $r = \frac{1}{100}$. It converges to $\frac{46/100}{1-1/100} = 46/99$.

64. $2.\overline{516} = 2 + \frac{516}{10^3} + \frac{2514}{10^6} + \cdots$. Now $\frac{516}{10^3} + \frac{2514}{10^6} + \cdots$ is a geometric series with $a = \frac{516}{10^3}$ and $r = \frac{1}{10^3}$. It converges to $\frac{a}{1-r} = \frac{516/10^3}{1-1/10^3} = \frac{516/10^3}{999/10^3} = \frac{516}{999}$. Thus, $2.\overline{516} = 2 + \frac{516}{999} = \frac{2514}{999} = \frac{838}{333}$.

68. $0.5\overline{29} = 0.5 + \frac{29}{10^3} + \frac{29}{10^5} + \frac{29}{10^7} + \cdots$. Now $\frac{29}{10^3} + \frac{29}{10^5} + \frac{29}{10^7} + \cdots$ is a geometric series with $a = \frac{29}{10^3}$ and $r = \frac{1}{10^2}$. It converges to $\frac{a}{1-r} = \frac{29/10^3}{1-1/10^2} = \frac{29}{990}$, so $0.5\overline{29} = 0.5 + \frac{29}{990} = \frac{262}{495}$, choice (**B**).

69. $\sum_{n=1}^{\infty}(-5)^n x^n = \sum_{n=1}^{\infty}(-5x)^n$ is a geometric series with $r = -5x$, so the series converges $\Leftrightarrow |r| < 1 \Leftrightarrow |-5x| < 1 \Leftrightarrow |x| < \frac{1}{5}$, that is, $-\frac{1}{5} < x < \frac{1}{5}$. In that case, the sum of the series is $\frac{-5x}{1-(-5x)} = \frac{-5x}{1+5x}$.

71. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n} = \sum_{n=0}^{\infty}\left(\frac{x-2}{3}\right)^n$ is a geometric series with $r = \frac{x-2}{3}$, so the series converges $\Leftrightarrow |r| < 1 \Leftrightarrow$ $\left|\frac{x-2}{3}\right| < 1 \Leftrightarrow -1 < \frac{x-2}{3} < 1 \Leftrightarrow -3 < x-2 < 3 \Leftrightarrow -1 < x < 5$. In that case, the sum of the series is $\frac{1}{1-\frac{x-2}{3}} = \frac{1}{\frac{3-(x-2)}{3}} = \frac{3}{5-x}$.

73. $\displaystyle\sum_{n=0}^{\infty}\frac{2^n}{x^n}=\sum_{n=0}^{\infty}\left(\frac{2}{x}\right)^n$ is a geometric series with $r=\dfrac{2}{x}$, so the series converges $\Leftrightarrow |r|<1 \Leftrightarrow \left|\dfrac{2}{x}\right|<1 \Leftrightarrow$ $2<|x| \Leftrightarrow x<-2$ or $x>2$. In that case, the sum of the series is $\dfrac{1}{1-2/x}=\dfrac{x}{x-2}$.

75. $\displaystyle\sum_{n=0}^{\infty}e^{nx}=\sum_{n=0}^{\infty}\left(e^x\right)^n$ is a geometric series with $r=e^x$, so the series converges $\Leftrightarrow |r|<1 \Leftrightarrow \left|e^x\right|<1 \Leftrightarrow$ $-1<e^x<1 \Leftrightarrow 0<e^x<1 \Leftrightarrow x<0$. In that case, the sum of the series is $\dfrac{1}{1-e^x}$.

78. Series I, $\displaystyle\sum_{n=1}^{\infty}\frac{8n}{n+8}$ diverges by the Test for Divergence because $\displaystyle\lim_{n\to\infty}\frac{8n}{n+8}=\lim_{n\to\infty}\frac{8}{1+8/n}=8\neq 0$. The series $\displaystyle\sum_{n=0}^{\infty}\left(\tfrac{1}{4}\right)^n$ is a geometric series with $r=0.25<1$, so it converges to $\dfrac{1}{1-1/4}=4/3$. Therefore, series II, $\displaystyle\sum_{n=0}^{\infty}\frac{6}{4^n}=6\cdot\sum_{n=0}^{\infty}\left(\tfrac{1}{4}\right)^n$ converges to $6\cdot\tfrac{4}{3}=8$. Series III, $\displaystyle\sum_{n=1}^{\infty}\frac{8}{n}=8\cdot\sum_{n=1}^{\infty}\frac{8}{n}$ is a constant multiple of the divergent harmonic series, so it diverges. Thus, the correct choice is (**B**), only series II converges to 8.

79. If $a>1$, then $a^{-1}=\dfrac{1}{a}<1\Rightarrow \displaystyle\sum_{n=0}^{\infty}a^{-n}=1+\sum_{n=1}^{\infty}\frac{1}{a^n}$. The series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{a^n}$ is geometric with $a=1$ and $r=a$, so its sum is $\dfrac{1}{1-a}$. Therefore, $\displaystyle\sum_{n=0}^{\infty}a^{-n}=1+\frac{1}{1-a}=\frac{1-a+1}{1-a}=\frac{-a}{1-a}=\frac{a}{a-1}$, choice (**A**).

80. If $x=\dfrac{2}{3}$, then $\displaystyle\sum_{n=0}^{\infty}\tfrac{1}{2}x\left(2x^2\right)^n=\sum_{n=0}^{\infty}\tfrac{1}{2}\tfrac{2}{3}\left(\tfrac{2\cdot 4}{9}\right)^n=\sum_{n=0}^{\infty}\tfrac{1}{3}\left(\tfrac{8}{9}\right)^n$ is geometric with $a=\tfrac{1}{3}$ and $r=\tfrac{8}{9}<1$, so it converges to $\dfrac{a}{1-r}=\dfrac{1/3}{1-8/9}=3$. If $x=-\dfrac{3}{4}$, then $\displaystyle\sum_{n=0}^{\infty}\tfrac{1}{2}x\left(2x^2\right)^n=\sum_{n=0}^{\infty}\tfrac{1}{2}\left(-\tfrac{3}{4}\right)\left(-\tfrac{2\cdot 3}{4}\right)^n=\sum_{n=0}^{\infty}-\tfrac{3}{8}\cdot\left(-\tfrac{3}{2}\right)^n$ is geometric with $r=-\tfrac{3}{2}$. Because $|r|=\tfrac{3}{2}>1$, this series diverges. Thus the correct choice is (**A**), $\tfrac{2}{3}$ only.

81. $\displaystyle\sum_{n=1}^{\infty}\frac{10\cdot 3^n+15\cdot 2^n}{5^n}=\sum_{n=1}^{\infty}\frac{10\cdot 3^n}{5^n}+\sum_{n=1}^{\infty}\frac{15\cdot 2^n}{5^n}=\sum_{n=1}^{\infty}10\cdot\left(\frac{3}{5}\right)^n+\sum_{n=1}^{\infty}15\cdot\left(\frac{2}{5}\right)^n$. The first series is a geometric series with $a=10\cdot\tfrac{3}{5}=6$ and $r=\tfrac{3}{5}=0.6<1$, so its sum is $\dfrac{a}{1-r}=\dfrac{6}{1-0.6}=15$. The second series is geometric with $a=15\cdot\tfrac{2}{5}=6$ and $r=\tfrac{2}{5}=0.4<1$, so its sum is $\dfrac{a}{1-r}=\dfrac{6}{1-0.4}=10$. Therefore, by Theorem 2, the sum of $\displaystyle\sum_{n=1}^{\infty}\frac{10\cdot 3^n+15\cdot 2^n}{5^n}$ is $15+10=25$, choice (**B**).

82. $\displaystyle\lim_{n\to\infty}\frac{en}{3n+1}=\lim_{n\to\infty}\frac{e}{3+1/n}=\frac{e}{n}\neq 0$, so series (**D**) diverges.