Calculators Allowed

1. B

2. a. 2.828 b. 11.588 c. $t = 2.208$, right d. 1, 3; .432, 1; 4

3. a. 1.145, 1.253 b. $y - 4.621 = .863(x - 9.315)$ c. 4.105

   d. (-28.425, 2.161)

4. a. 47.513 b. $\theta = 2.017$, $y = 6.272$ c. -2.819, y dec.

5. 10.370

No Calculators

1. B 2. C 3. E 4. A 5. D 6. D 7. $x^2 + y^2 = 3x$

8. a. $\frac{1}{2}\int_0^{\pi/3}\left(1-2\cos\theta\right)^2 d\theta$ b. $\frac{dx}{d\theta} = 4\sin\theta\cos\theta - \sin\theta$

$$\frac{dy}{d\theta} = \cos\theta - 2\cos^2\theta + 2\sin^2\theta$$

c. $y - 1 = -2x$

Name ________________

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Calculus BC– Chapter 10 Sample Test (calculators allowed)

Show all work for free-response questions.

1. A polar curve is given by $r = \frac{3}{2-\cos\theta}$. The slope of the curve at $\theta = \frac{\pi}{2}$ is

(A) 0 (B) 0.5 (C) 0.75 (D) -0.75

$x = \frac{3}{2-\cos\theta}\cos\theta$ $y = \frac{3}{2-\cos\theta}\sin\theta$ $m = \frac{y'(\pi/2)}{x'(\pi/2)} = .5$

$x'(\frac{\pi}{2}) = -1.5$ $y'(\frac{\pi}{2}) = -.75$

2. A particle is moving along a curve so that its position at time t is $(x(t), y(t))$, where $x(t) = t^2 - 4t + 8$ and $y(t)$ is not explicitly given. Both x and y are measured in meters, and t is measured in seconds. It is known that $\frac{dy}{dt} = te^{t-3} - 1$.

a. Find the speed of the particle at time $t = 3$ seconds.

$x'(t) = 2t - 4$ speed $= \sqrt{(2t-4)^2 + (te^{t-3}-1)^2} \xrightarrow{\text{at } t=3} \boxed{2.828}$

b. Find the total distance traveled by the particle for $0 \le t \le 4$ seconds.

$\int_0^4 \sqrt{(2t-4)^2 + (te^{t-3}-1)^2}\,dt = 11.588$

c. Find the time t, $0 \le t \le 4$, when the line tangent to the path of the particle is horizontal. Is the direction of the motion of the particle toward the left or toward the right at that time? Give a reason for your answer.

horiz $\Rightarrow \frac{dy}{dt} = 0$ $\frac{dx}{dt}(2.208) = .416$

$t = 2.208$ right because $\frac{dx}{dt} > 0$ at the point

d. There is a point with x-coordinate 5 through which the particle passes twice. Find the following:

(i) The two values of t when that occurs.

$t^2 - 4t + 8 = 5$

$t^2 - 4t + 3 = 0$

$(t-1)(t-3) = 0$ $\rightarrow t = 1, 3$

(ii) The slopes of the lines tangent to the particle's path at that point.

$\frac{dy}{dx} = \frac{te^{t-3} - 1}{2t - 4}$ $\left.\frac{dy}{dx}\right|_{t=1} = .432$ $\left.\frac{dy}{dx}\right|_{t=3} = 1$

(iii) The y-coordinate of that point, given $y(2) = 3 + \frac{1}{e}$.

$y(3) = y(2) + \int_2^3 (te^{t-3} - 1)\,dt = 4$

3. The velocity vector of a particle moving in the xy-plane has components given by $\frac{dx}{dt} = 14\cos(t^2)\sin(e^t)$ and $\frac{dy}{dt} = 1 + 2\sin(t^2)$ for $0 \le t \le 1.5$. At time $t = 0$, the position of the particle is $(-2,3)$.

a. For $0 < t < 1.5$, find all values of t at which the line tangent to the path of the particle is vertical.

vert. $\Rightarrow \frac{dy}{dx}$ undef. $\Rightarrow \frac{dx}{dt} = 0$

$t = 1.145,\ 1.253$

b. Write an equation for the line tangent to the path of the particle at $t = 1$.

$x(1) = -2 + \int_0^1 14\cos(t^2)\sin(e^t)\,dt = 9.315$

$y(1) = 3 + \int_0^1 1 + 2\sin(t^2)\,dt = 4.621$

$m = \left.\frac{1 + 2\sin(t^2)}{14\cos(t^2)\sin(e^t)}\right|_{t=1} = .863$

$\boxed{y - 4.621 = .863(x - 9.315)}$

c. Find the speed of the particle at $t = 1$.

$\left.\sqrt{(14\cos(t^2)\sin(e^t))^2 + (1+\sin(t^2))^2}\right|_{t=1} = 4.105$

d. Find the acceleration vector of the particle at $t = 1$.

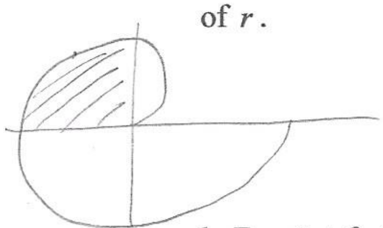
$\left.\frac{d}{dt}(14\cos(t^2)\sin(e^t))\right|_{t=1} = -28.425$

$\left.\frac{d}{dt}(1 + 2\sin(t^2))\right|_{t=1} = 2.161$

$\boxed{\langle -28.425, 2.161 \rangle}$

4. The polar curve r is given by $r(\theta)=3\theta+\sin\theta$, where $0\le\theta\le 2\pi$.

a. Find the area in the second quadrant enclosed by the coordinate axes and the graph of r.



$\frac{1}{2}\int_{\pi/2}^{\pi}(3\theta+\sin\theta)^2\,d\theta = 47.513$

b. For $\frac{\pi}{2}\le\theta\le\pi$, there is one point P on the polar curve r with x-coordinate -3. Find the angle θ that corresponds to point P. Find the y-coordinate of point P. Show the work that leads to your answers.

$x=(3\theta+\sin\theta)\cos\theta=-3$

$\theta=2.017$

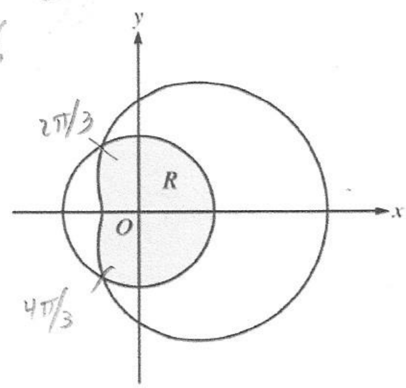
$y=(3\theta+\sin\theta)\sin\theta \xrightarrow{\text{at } \theta=2.017} y=6.272$

c. A particle is traveling along the polar curve r so that its position at time t is $(x(t),y(t))$ and such that $\frac{d\theta}{dt}=2$. Find $\frac{dy}{dt}$ at the instant that $\theta=\frac{2\pi}{3}$, and interpret the meaning of your answer in the context of the problem.

$y=(3\theta+\sin\theta)\sin\theta$

$\frac{dy}{dt}=\left.\frac{dy}{d\theta}\right|_{\theta=2\pi/3}\cdot 2=-2.819$

When $\theta=\frac{2\pi}{3}$, y-coord. is dec. at a rate of 2.819.



5. The graphs of the polar curves $r=2$ and $r=3+2\cos\theta$ are shown in the figure above. The curves intersect when $\theta=\frac{2\pi}{3}$ and $\theta=\frac{4\pi}{3}$. Let R be the region that is inside the graph of $r=2$ and also inside the graph $r=3+2\cos\theta$, as shaded in the figure. Find the area of R.

$A=2\left[\frac{1}{2}\int_{0}^{2\pi/3}(2)^2\,d\theta+\frac{1}{2}\int_{2\pi/3}^{\pi}(3+2\cos\theta)^2\,d\theta\right]=10.370$

Name ___________________

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Calculus BC– Chapter 10 Sample Test (no calculators)

Show all work for free-response questions.

1. At time $t \geq 0$, a particle moving in the xy-plane has velocity vector given by $v(t) = \langle t^2, 5t \rangle$. What is the acceleration vector of the particle at time $t = 3$?

   (A) $\langle 9, \frac{45}{2} \rangle$ (B) $\langle 6, 5 \rangle$ (C) $\langle 2, 0 \rangle$ (D) $\sqrt{306}$ (E) $\sqrt{61}$

   $a(t) = \langle 2t, 5 \rangle$
   $a(3) = \langle 6, 5 \rangle$

2. The position of a particle moving in the xy-plane is given by the parametric equations $x = t^3 - 3t^2$ and $y = 2t^3 - 3t^2 - 12t$. For what values of t is the particle at rest?

   (A) -1 only (B) 0 only (C) 2 only

   (D) -1 and 2 only (E) -1, 0, and 2

   $\frac{dx}{dt} = 3t^2 - 6t = 0$
   $3t(t-2) = 0$
   $t = 0, 2$

   $\frac{dy}{dt} = 6t^2 - 6t - 12$
   $= 6(t^2 - t - 2)$
   $= 6(t-2)(t+1) = 0$
   $t = -1, 2$

3. If $x = e^{2t}$ and $y = \sin(2t)$, then $\frac{dy}{dx} =$

   (A) $4e^{2t}\cos(2t)$ (B) $\frac{e^{2t}}{\cos(2t)}$ (C) $\frac{\sin(2t)}{2e^{2t}}$

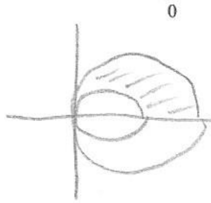
   (D) $\frac{\cos(2t)}{2e^{2t}}$ (E) $\frac{\cos(2t)}{e^{2t}}$

   $\frac{dy}{dx} = \frac{2\cos(2t)}{2e^{2t}}$

4. Which of the following is equal to the area of the region inside the polar curve $r = 2\cos\theta$ and outside the polar curve $r = \cos\theta$?

   (A) $3\int_0^{\pi/2} \cos^2\theta \, d\theta$ (B) $3\int_0^{\pi} \cos^2\theta \, d\theta$ (C) $\frac{3}{2}\int_0^{\pi/2} \cos^2\theta \, d\theta$

   (D) $3\int_0^{\pi/2} \cos\theta \, d\theta$ (E) $3\int_0^{\pi} \cos\theta \, d\theta$



$\leftarrow \frac{1}{2}\int_0^{\pi/2} (2\cos\theta)^2 - (\cos\theta)^2 \, d\theta = \frac{1}{2}\int_0^{\pi/2} 3\cos^2\theta \, d\theta$

$\rightarrow$ double it

5. The length of the path described by the parametric equations $x = \cos^3 t$ and $y = \sin^3 t$, for $0 \le t \le \frac{\pi}{2}$, is given by

$x' = 3\cos^2 t(-\sin t)$

$y' = 3\sin^2 t \cos t$

(A) $\int_0^{\pi/2} \sqrt{3\cos^2 t + 3\sin^2 t}\, dt$

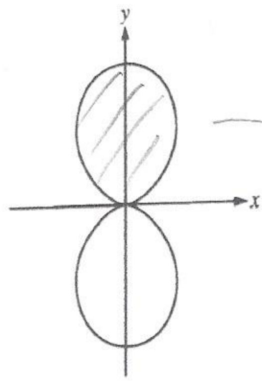
(B) $\int_0^{\pi/2} \sqrt{-3\cos^2 t \sin t + 3\sin^2 t \cos t}\, dt$

(C) $\int_0^{\pi/2} \sqrt{9\cos^4 t + 9\sin^4 t}\, dt$

(D) $\int_0^{\pi/2} \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t}\, dt$

(E) $\int_0^{\pi/2} \sqrt{\cos^6 t + \sin^6 t}\, dt$

$S = \int_0^{\pi/2} \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t}\, dt$



$\to \frac{1}{2}\int_0^{\pi} [\sin^2\theta]^2\, d\theta$

$\to$ double it

6. Which of the following expressions gives the total area enclosed by the polar curve $r = \sin^2 \theta$ shown in the figure above?

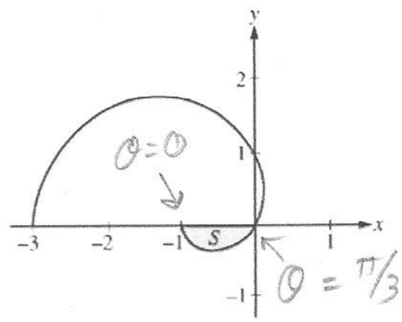
(A) $\frac{1}{2}\int_0^{\pi} \sin^2 \theta\, d\theta$ (B) $\int_0^{\pi} \sin^2 \theta\, d\theta$ (C) $\frac{1}{2}\int_0^{\pi} \sin^4 \theta\, d\theta$

(D) $\int_0^{\pi} \sin^4 \theta\, d\theta$ (E) $2\int_0^{\pi} \sin^4 \theta\, d\theta$

7. Convert the polar equation $r = 3\cos\theta$ to rectangular form.

$r^2 = 3r\cos\theta$

$\boxed{x^2 + y^2 = 3x}$



$1 - 2\cos\theta = 0$
$\cos\theta = \frac{1}{2}$
$\theta = \pi/3$

8. The graph of the polar curve $r = 1 - 2\cos\theta$ for $0 \le \theta \le \pi$ is shown above. Let S be the shaded region in the third quadrant bounded by the curve and the x-axis.

a. Write an integral expression for the area of S. DO NOT EVALUATE.

$$\frac{1}{2}\int_0^{\pi/3} [1 - 2\cos\theta]^2 \, d\theta$$

b. Write expressions for $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ in terms of θ.

$x = (1 - 2\cos\theta)\cos\theta$

$\frac{dx}{d\theta} = (1 - 2\cos\theta)(-\sin\theta) + \cos\theta(2\sin\theta) = 4\sin\theta\cos\theta - \sin\theta$

$y = (1 - 2\cos\theta)\sin\theta$

$\frac{dy}{d\theta} = (1 - 2\cos\theta)(\cos\theta) + \sin\theta(2\sin\theta) = \cos\theta - 2\cos^2\theta + 2\sin^2\theta$

c. Write an equation in terms of x and y for the line tangent to the graph of the polar curve at the point where $\theta = \frac{\pi}{2}$. Show the computations that lead to your answer.

$x\left(\frac{\pi}{2}\right) = \left(1 - 2\cos\frac{\pi}{2}\right)\cos\frac{\pi}{2} = 0$

$y\left(\frac{\pi}{2}\right) = \left(1 - 2\cos\frac{\pi}{2}\right)\sin\frac{\pi}{2} = 1$

$y - 1 = -2(x - 0)$

$m = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{\pi/2} = \frac{\cos\frac{\pi}{2} - 2\cos^2\frac{\pi}{2} + 2\sin^2\frac{\pi}{2}}{4\sin\frac{\pi}{2}\cos\frac{\pi}{2} - \sin\frac{\pi}{2}} = \frac{2}{-1} = -2$