Review, Part 3

<u>Average Value</u>

$$\begin{pmatrix}\text{ave. value of}\\ f(x)\text{ on }[a,b]\end{pmatrix} = \frac{1}{b-a}\int_a^b f(x)dx$$

<u>Ex.</u> Find the average value of $f(x) = x^2\sqrt{x^3+1}$ on $[0,2]$.

$$\frac{1}{2}\int_0^2 x^2\sqrt{x^3+1}\,dx = \frac{1}{2}\int_*^{**} u^{1/2}\cdot\frac{1}{3}du = \frac{1}{6}\cdot\frac{2}{3}u^{3/2}\Big|_*^{**} = \frac{1}{9}(x^3+1)^{3/2}\Big|_0^2$$

$$= \frac{1}{9}(9)^{3/2} - \frac{1}{9}(1)^{3/2} = \frac{1}{9}(27-1) = \boxed{\frac{26}{9}}$$

$u = x^3+1$
$du = 3x^2dx$
$\frac{1}{3}du = x^2dx$

Remember:

$$\begin{pmatrix}\text{ave. value of}\\ f(x) \text{ on } [a,b]\end{pmatrix} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\begin{pmatrix}\text{ave. rate of change}\\ \text{from } t=a \text{ to } t=b\end{pmatrix} = \frac{f(b)-f(a)}{b-a} = f'(c)$$

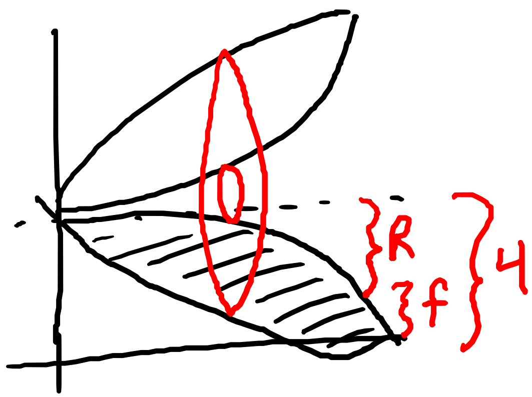
Volumes

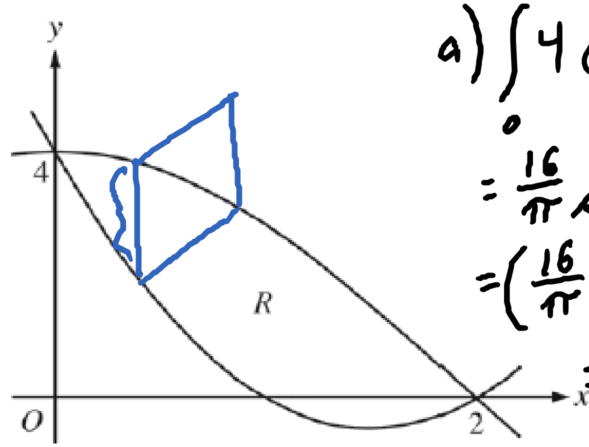
Rotation: $$V = \pi \int_a^b \left[R^2 - r^2 \right] dx$$

→ dx or dy depending on the axis of rotation

Cross-section: $$V = \int_a^b A(x)dx$$

2013 AB#5





a) $\int_0^2 4\cos\left(\frac{\pi}{4}x\right) - (2x^2-6x+4)\,dx$

$= \frac{16}{\pi}\sin\left(\frac{\pi}{4}x\right) - \frac{2}{3}x^3 + 3x^2 - 4x \Big|_0^2$

$= \left(\frac{16}{\pi}\sin\frac{\pi}{2} - \frac{2}{3}\cdot 8 + 3\cdot 4 - 4\cdot 2\right) - 0$

$= \frac{16}{\pi} - \frac{16}{3} + 4$

$R = 4 - f(x)$

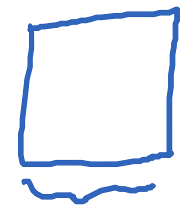
$r = 4 - g(x)$

Let $f(x) = 2x^2 - 6x + 4$ and $g(x) = 4\cos\left(\frac{1}{4}\pi x\right)$. Let R be the region bounded by the graphs of f and g, as shown in the figure above.

(a) Find the area of R.

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 4$.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Write, but do not evaluate, an integral expression that gives the volume of the solid.



$g(x)-f(x)$

b) $V = \pi\int_0^2 (4-f(x))^2 - (4-g(x))^2\,dx$

c) $V = \int_0^2 (g(x)-f(x))^2\,dx$

Ex. Find the length of $y = \frac{x^3}{6} + \frac{1}{2x}$ on [1,2].

$$y = \frac{1}{6}x^3 + \frac{1}{2}x^{-1}$$

$$y' = \frac{1}{2}x^2 - \frac{1}{2}x^{-2}$$

$$S = \int_a^b \sqrt{1 + (y')^2}\, dx$$

$$= \int_1^2 \sqrt{1 + \left(\frac{1}{2}x^2 - \frac{1}{2}x^{-2}\right)^2}\, dx$$

Integration

Ex. $\int x \sec^2 x \, dx$

$$\boxed{\begin{array}{ll} u = x & dv = \sec^2 x\, dx \\ du = dx & v = \tan x \end{array}}$$

$= x\tan x - \int \tan x \, dx$

$= x\tan x - \ln|\sec x| + C$

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u-sub.

parts

~~trig. int.~~

~~trig. subst.~~

partial fract.

$uv - \int v\, du$

Ex. $\int \frac{1}{x^2-5x+6}\,dx = \int \frac{-1}{x-2} + \frac{1}{x-3}\,dx$

$(x-2)(x-3)$

$= -\ln|x-2| + \ln|x-3| + C$

$= \ln\left|\frac{x-3}{x-2}\right| + C$

$\frac{1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$

$1 = A(x-3) + B(x-2)$

$\underline{x=3:}$ $1 = B(1) \rightarrow B = 1$

$\underline{x=2:}$ $1 = A(-1) \rightarrow A = -1$

Ex. $\int_1^\infty (1-x)e^{-x}dx = \lim_{R\to\infty} \int_1^R (1-x)e^{-x}dx = \lim_{R\to\infty} xe^{-x}\Big|_1^R$

$$= \lim_{R\to\infty}\left(\frac{R}{e^R} - \frac{1}{e^1}\right) = 0 - \frac{1}{e}$$

$$\lim_{R\to\infty}\frac{R}{e^R} \overset{L}{=} \lim_{R\to\infty}\frac{1}{e^R} = 0$$

$$\lim_{R\to\infty} R = \infty \qquad \lim_{R\to\infty} e^R = \infty$$

$$\int (1-x)e^{-x}dx = (1-x)(-e^{-x}) - \int +e^{-x}(+1)dx = -e^{-x} + xe^{-x} + e^{-x} = xe^{-x}$$

$$\boxed{\begin{array}{ll} u = 1-x & dv = e^{-x} \\ du = -dx & v = -e^{-x} \end{array}}$$

Ex. Determine the convergence of $\sum_{n=1}^{\infty} \frac{(-1)^n (n+1)}{n^3+2} \rightarrow$

$$\sum \frac{n+1}{n^3+2}$$

$$\downarrow$$

$$\sum \frac{n}{n^3} = \sum \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{\frac{n+1}{n^3+2}}{1/n^2} = \lim_{n\to\infty} \frac{n+1}{n^3+2} \cdot \frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{n^3+\cancel{n^2}}{n^3+\cancel{2}} = 1$$

$\sum \frac{1}{n^2}$ conv. by p-series, $p=2$

$\therefore \sum \frac{n+1}{n^3+2}$ conv. by limit. comp. test

$\therefore \sum \frac{(-1)^n(n+1)}{n^3+2}$ conv. by abs. conv. test

Ex. Determine the convergence of $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^3}} = \sum \frac{1}{n^{3/4}}$

div. by p-series test, $p = \frac{3}{4}$

Ex. Determine the convergence of $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+1}$

$\sum \frac{n}{n+1}$

$\lim_{n\to\infty} \frac{(-1)^n n}{n+1} \neq 0$

$\therefore$ div. by Test for Div.

Ex. Determine the convergence of $\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$

$$\int_1^\infty \frac{x}{e^{x^2}}\,dx = \lim_{R\to\infty}\int_1^R x e^{-x^2}\,dx = \lim_{R\to\infty} -\frac{1}{2}e^{-x^2}\Big|_1^R$$

$$= \lim_{R\to\infty}\left(\underbrace{\frac{-1}{2e^{R^2}}}_{0} + \frac{1}{2e^{1}}\right) = \frac{1}{2e}$$

cont. ✓
pos. ✓
dec. ✓

$\therefore \sum \frac{n}{e^{n^2}}$ conv. by Integral Test

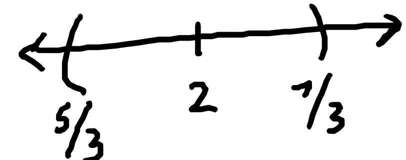
Ex. Find the interval of convergence for

$$\sum_{n=1}^{\infty}\frac{3^n(x-2)^n}{n}$$

$$\lim_{n\to\infty}\left|\frac{\cancel{3^{n+1}}\,\cancel{(x-2)^{n+1}}}{n+1}\cdot\frac{n}{\cancel{3^n}\,\cancel{(x-2)^n}}\right| = \lim_{n\to\infty}\left|\underbrace{\frac{n}{n+1}}_{\to 1}\cdot 3(x-2)\right| = |3(x-2)| < 1$$

$$|x-2| < \frac{1}{3}$$

$\underline{x=\frac{7}{3}:}$ $\displaystyle\sum\frac{3^n\left(\frac{7}{3}-2\right)^n}{n} = \sum\frac{3^n\left(\frac{1}{3}\right)^n}{n} = \sum\frac{1}{n}$ div. by harm.



$\underline{x=\frac{5}{3}:}$ $\displaystyle\sum\frac{3^n\left(\frac{5}{3}-2\right)^n}{n} = \sum\frac{3^n\left(-\frac{1}{3}\right)^n}{n} = \sum\frac{(-1)^n}{n}$ conv. by alt. harm.

$$\boxed{\left[\frac{5}{3},\frac{7}{3}\right)}$$

2013 #6

A function f has derivatives of all orders at $x = 0$. Let $P_n(x)$ denote the nth-degree Taylor polynomial for f about $x = 0$.

(a) It is known that $f(0) = -4$ and that $P_1\left(\frac{1}{2}\right) = -3$. Show that $f'(0) = 2$.

$$P_1(x) = f(c) + f'(c)(x-c)$$
$$= f(0) + f'(0)x$$
$$= -4 + f'(0)x$$
$$P_1\left(\tfrac{1}{2}\right) = -4 + f'(0)\left(\tfrac{1}{2}\right) = -3$$
$$f'(0) = 2$$

(b) It is known that $f''(0) = -\frac{2}{3}$ and $f'''(0) = \frac{1}{3}$. Find $P_3(x)$.

(c) The function h has first derivative given by $h'(x) = f(2x)$. It is known that $h(0) = 7$. Find the third-degree Taylor polynomial for h about $x = 0$.

b) $$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$
$$= -4 + 2x + \frac{-2/3}{2}x^2 + \frac{1/3}{6}x^3 \longrightarrow f(x) \approx -4 + 2x - \frac{1}{3}x^2 + \frac{1}{18}x^3$$

c) $$h'(x) = f(2x) \approx -4 + 2(2x) - \frac{1}{3}(2x)^2 + \frac{1}{18}(2x)^3 = -4 + 4x - \frac{4}{3}x^2 + \frac{4}{9}x^3$$

$$h(x) \approx 7 - 4x + 2x^2 - \frac{4}{9}x^3$$