- Blue part is out of 55
- Green part is out of 45

Review, Part 1

To differentiate an implicit function, we differentiate term-by-term:

- Take the derivative of x-function as usual.
- The derivative of y-function gets multiplied by y'.
- If x's and y's are in the same term, use product rule.

After differentiating, solve for y'.

Ex. $x^2 + \underline{4xy^3} - y^4 = 17$, find the equation of the tangent line at (1,2).

$$2x + \underline{4x \cdot 3y^2 y' + y^3 \cdot 4} - 4y^3 y' = 0$$

$$2(1) + 4(1)3(2)^2 y' + (2)^3 \cdot 4 - 4(2)^3 y' = 0$$

$$2 + 48y' + 32 - 32y' = 0$$

$$16y' = -34$$

$$y' = -\frac{17}{8}$$

$$\boxed{y - 2 = -\frac{17}{8}(x-1)}$$

Ex. Let $f(x) = x^3 + x$. If $g(x) = f^{-1}(x)$ and $f(2) = 10$, find $g'(10)$.

f: $y = x^3 + x$ $(2, 10)$

g: $x = y^3 + y$ $(10, 2)$

$1 = 3y^2 y' + 1 \cdot y'$

$1 = 3(2)^2 y' + y'$

$1 = 13y'$ $\rightarrow$ $y' = \frac{1}{13}$

2013 #5

Consider the differential equation $\frac{dy}{dx} = y^2(2x+2)$. Let $y = f(x)$ be the particular solution to the differential equation with initial condition $f(0) = -1$.

(a) Find $\lim_{x\to 0} \frac{f(x)+1}{\sin x}$. Show the work that leads to your answer.

(b) Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f\left(\frac{1}{2}\right)$.

(c) Find $y = f(x)$, the particular solution to the differential equation with initial condition $f(0) = -1$.

a) $\lim_{x\to 0} \frac{f(x)+1}{\sin x} \overset{L}{=} \lim_{x\to 0} \frac{f'(x)}{\cos x} = \frac{(-1)^2(2\cdot 0+2)}{1} = 2$

$\lim_{x\to 0}(f(x)+1) = 0$

$\lim_{x\to 0} \sin x = 0$

b)

Δx	x_1	y_1	$m = y^2(2x+2)$	$y_2 = y_1 + m\cdot\Delta x$
$1/4$	0	-1	2	$-1+2\left(\frac{1}{4}\right) = -\frac{1}{2}$
$1/4$	$1/4$	$-\frac{1}{2}$	$\left(-\frac{1}{2}\right)^2\left(\frac{1}{2}+2\right) = \frac{5}{8}$	$-\frac{1}{2}+\frac{5}{8}\left(\frac{1}{4}\right) = \frac{-11}{32}$
$1/4$	$1/2$	$\frac{-11}{32}$		

$f\left(\frac{1}{2}\right) \approx \frac{-11}{32}$

c) $\frac{dy}{dx} = y^2(2x+2)$

$\int \frac{1}{y^2}\,dy = \int (2x+2)\,dx$

$-\frac{1}{y} = x^2+2x+C$

$y = \frac{1}{-x^2-2x+D}$

$-1 = \frac{1}{0+D}$

$D = -1$

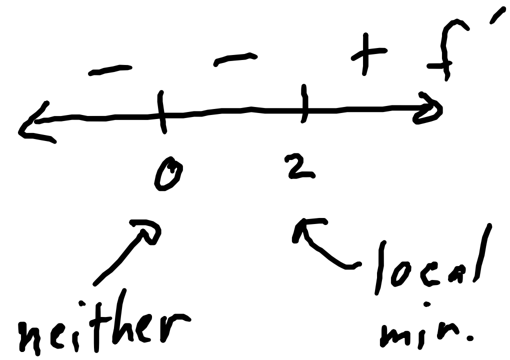
$$\boxed{y = \frac{1}{-x^2-2x-1}}$$

Critical Points

→ Where $f' = 0$ or is undefined

Ex. If $f(x) = x^4 - \frac{8}{3}x^3$, find and classify all critical points

$$f' = 4x^3 - 8x^2$$
$$= 4x^2(x-2) = 0$$
$$\boxed{x=0} \quad \boxed{x=2}$$



→ Inflection points are where the concavity of f changes

<u>Ex.</u> If $f(x) = x^4 - \frac{8}{3}x^3$, find all inflection points.

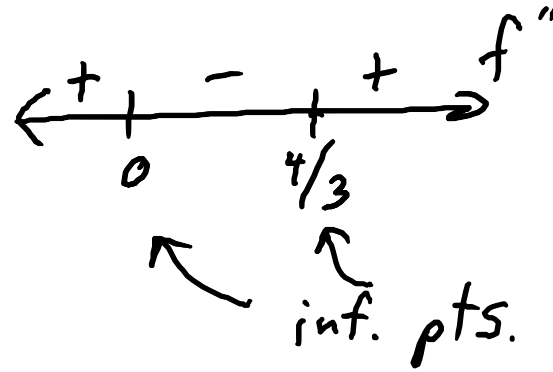
Where is f concave up?

$f' = 4x^3 - 8x^2$

$f'' = 12x^2 - 16x$

$= 4x(3x-4) = 0$

$x = 0 \quad x = \frac{4}{3}$



$\boxed{(-\infty, 0) \quad (\tfrac{4}{3}, \infty)}$

Absolute max/min

→ Check all critical points and endpoints

Ex. If the velocity of a particle is given by $v(t) = t^3 - 3t^2 + 12t + 4$, find its maximum acceleration on the interval $0 \leq t \leq 3$.

$a(t) = 3t^2 - 6t + 12$

$a'(t) = 6t - 6 = 0$

$t = 1$

$a(0) = 0 + 0 + 12 = 12$

$a(1) = 3 - 6 + 12 = 9$

$a(3) = 27 - 18 + 12 = 21$

$\boxed{\text{max. accel. is } 21}$

2013 #4

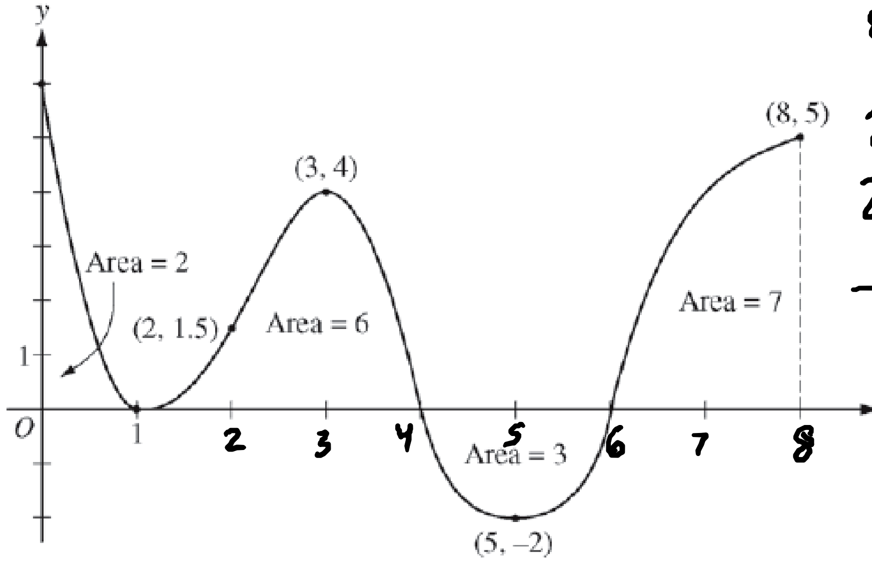
$f(0) = -8$

$\cancel{f(8)}$ = f' pos. before

$\cancel{f(1)}$ = not max. or min.

$\cancel{f(4)}$ = local max.

$f(6) = -3$



Graph of f'

$\int_0^8 f'(x) = f(8) - f(0)$

$2 + 6 - 3 + 7 = 4 - f(0)$

$f(0) = -8$

$\int_6^8 f'(x)\,dx = f(8) - f(6)$

$7 = 4 - f(6)$

$f(6) = -3$

The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the closed interval $0 \le x \le 8$. The graph of f' has horizontal tangent lines at $x = 1$, $x = 3$, and $x = 5$. The areas of the regions between the graph of f' and the x-axis are labeled in the figure. The function f is defined for all real numbers and satisfies $f(8) = 4$.

(a) Find all values of x on the open interval $0 < x < 8$ for which the function f has a local minimum. Justify your answer.

$x = 6$, f' goes neg. to pos.

(b) Determine the absolute minimum value of f on the closed interval $0 \le x \le 8$. Justify your answer.

-8

(c) On what open intervals contained in $0 < x < 8$ is the graph of f both concave down and increasing? Explain your reasoning.

$(0,1), (3,4)$

f' dec. f' pos.

(d) The function g is defined by $g(x) = (f(x))^3$. If $f(3) = -\frac{5}{2}$, find the slope of the line tangent to the graph of g at $x = 3$.

$g(x) = (f(x))^3$

$g'(x) = 3(f(x))^2 f'(x)$

$g'(3) = 3(f(3))^2 f'(3)$

$= 3\left(\frac{-5}{2}\right)^2 (4)$

$= 75$

Ex. A particle moves along a curve defined by the equation $2x^2 + 3y^2 - 4xy = 36$. At time $t = 1$, the particle is at the point $(2, -2)$ and the rate of change of the y-coordinate is 4. Find the rate of change of the x-coordinate at $t = 1$.

$\frac{dy}{dt}$ $\frac{dx}{dt}$

$$2x^2 + 3y^2 \underbrace{-4xy} = 36$$

$$4x\frac{dx}{dt} + 6y\frac{dy}{dt} + (-4x)\frac{dy}{dt} + y(-4)\frac{dx}{dt} = 0$$

$$4(2)\frac{dx}{dt} + 6(-2)(4) + (-4)(2)(4) + (-2)(-4)\frac{dx}{dt} = 0$$

$$8\frac{dx}{dt} - 48 - 32 + 8\frac{dx}{dt} = 0$$

$$16\frac{dx}{dt} = 80$$

$$\frac{dx}{dt} = 5$$

Ex. Let f be a twice-differentiable function such that $f(0) = -13$ and $f(7) = 15$. Must there exist a value of c, for $0 < c < 7$, such that $f(c) = 0$? Justify your answer.

$f(0) < 0$ $\quad \therefore\ f(c) = 0$ on interval by IVT

$f(7) > 0$

f is cont. because f is twice diff.

- You need to state that f is continuous to use IVT
- You need to state how you know that f is continuous
- You need to state that the desired value is between the two given values

2013 #1

On a certain workday, the rate, in tons per hour, at which unprocessed gravel arrives at a gravel processing plant is modeled by $G(t) = 90 + 45\cos\left(\frac{t^2}{18}\right)$, where t is measured in hours and $0 \le t \le 8$. At the beginning of the workday $(t = 0)$, the plant has 500 tons of unprocessed gravel. During the hours of operation, $0 \le t \le 8$, the plant processes gravel at a constant rate of 100 tons per hour.

(a) Find $G'(5)$. Using correct units, interpret your answer in the context of the problem.

(b) Find the total amount of unprocessed gravel that arrives at the plant during the hours of operation on this workday.

(c) Is the amount of unprocessed gravel at the plant increasing or decreasing at time $t = 5$ hours? Show the work that leads to your answer.

(d) What is the maximum amount of unprocessed gravel at the plant during the hours of operation on this workday? Justify your answer.

$G \rightarrow \frac{\text{tons}}{\text{hr.}}$

$\frac{dG}{dt} \rightarrow \frac{\text{tons/hr}}{\text{hr}}$

d) $G(t) - 100 = 0$

$t = 4.923$

$A(0) = 500$

$A(4.923) = 500 + \int_0^{4.923} G - 100\, dt$

$= 635.376$

$A(8) = 525.551$

a) $G'(5) = -24.588$ tons/hr²; 5 hours after work starts, the rate at which gravel enters is decreasing at a rate of 24.588 tons/hr².

b) $\int_0^8 G(t)\,dt = 825.551$

c) $A(t) = 500 + \int_0^t (G(x) - 100)\,dx$

$A'(t) = G(t) - 100 \rightarrow A'(5) = G(5) - 100 = -1.859$ dec. because $A'(5) < 0$