Alternating Series

An alternating series is a series where terms alternate in sign.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Thm. Alternating Series Test for Convergence

The alternating series $\sum(-1)^n a_n$ will converge if

i) $\lim_{n\to\infty} a_n = 0$

ii) a_n is a decreasing sequence
($a_{n+1} < a_n$ after some value of n)

→ Be sure to verify conditions before using this test.

Ex. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

i) $\lim_{n\to\infty} \frac{1}{n} = 0$ ✓

ii) dec. : $\frac{1}{n+1} < \frac{1}{n}$ ✓

$\therefore \sum \frac{(-1)^n}{n}$ conv. by Alt. Series Test for Conv.

Ex. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$

i) $\lim_{n\to\infty} \frac{1}{n!} = 0$ ✓

ii) dec.: $\frac{1}{(n+1)!} < \frac{1}{n!}$ ✓

$\therefore \sum \frac{(-1)^{n-1}}{n!}$ conv. by Alt. Series test for Conv.

Thm. Absolute Convergence Test

If $\sum |a_n|$ converges, then $\sum a_n$ converges.

Ex. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$

$\sum \frac{1}{n^3}$ conv. by p-series test, $p=3$

$\therefore \sum \frac{(-1)^n}{n^3}$ conv. by Abs. Conv. Test

$\sum a_n$ is <u>absolutely convergent</u> if $\sum |a_n|$ and $\sum a_n$ both converge.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$$

$\sum a_n$ is <u>conditionally convergent</u> if $\sum a_n$ converges but $\sum |a_n|$ diverges.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

Ex. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^n+5}$ $\longrightarrow$ $\displaystyle\sum \frac{1}{2^n+5}$

compare

$\displaystyle\sum \frac{1}{2^n}$

$$\lim_{n\to\infty} \frac{\frac{1}{2^n+5}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{2^n}{2^n+5} = 1$$

$\sum \frac{1}{2^n}$ conv. by Geom. Series Test, $r = \frac{1}{2}$

$\therefore \sum \frac{1}{2^n+5}$ conv. by Limit Comp. Test

$\therefore \sum \frac{(-1)^{n-1}}{2^n+5}$ abs. conv. by Abs. Conv. Test

Checking Convergence of $\sum(-1)^n a_n$

Determine convergence of $\sum a_n$

$\sum a_n$ convergent

$\sum a_n$ divergent

$\sum(-1)^n a_n$ abs. conv. by Abs. Conv. Test

Check for cond. conv. of $\sum(-1)^n a_n$ using Alt. Series Test

$\underline{\text{Ex.}}$ $$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \longrightarrow \sum \frac{1}{\sqrt{n}}$$

div. by p-series test, $p = \frac{1}{2}$

i) $\lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0$ ✓

ii) $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$ ✓

$\therefore \sum \frac{(-1)^n}{\sqrt{n}}$ is cond. conv. by Alt. Series Test for Conv.

Ex. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2} \longrightarrow \sum \frac{|\cos n|}{n^2}$

conv. by direct comp. Test (see previous lesson)

$\therefore \sum \frac{\cos n}{n^2}$ conv. by Abs. Conv. Test

This is not alternating, but…

Thm. Alternating Series Remainder

If you use N terms to approximate the sum of the convergent alternating series $\sum(-1)^n a_n$, then the error is less than a_{N+1}.

Ex. Approximate the value of $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$ $(-1)^n$

using the first 6 terms. Show that this approx. differs from the exact value by less than $\frac{3}{20}$.

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n} \approx \frac{-1}{1} + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6}$$

$$\text{error} < \frac{1}{7} < \frac{3}{20}$$

✓

Pract.

Determine if the series is absolutely convergent, conditionally convergent, or divergent. State the test used.

1. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+1}$ Cond. Conv.; harm., LCT, and Alt Series Test for Conv.

2. $\sum_{n=1}^{\infty}\frac{(-1)^n 3^n}{4^n-1}$ Abs. Conv.; Geom., LCT, and Abs. Conv. Test

T-shirt design ideas are due next class.