Ex. $\int_{1}^{3} x^3 dx = \frac{1}{4}x^4\Big|_1^3 = \frac{1}{4}3^4 - \frac{1}{4}1^4$

Thm. Fundamental Theorem of Calculus

If $f(x)$ is a continuous function on $[a,b]$, and if $F'(x) = f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Improper Integrals

An improper integral is one where an endpoint is infinite or where there is a vertical asymptote on the interval.

→ To evaluate, we must change these problems to limits.

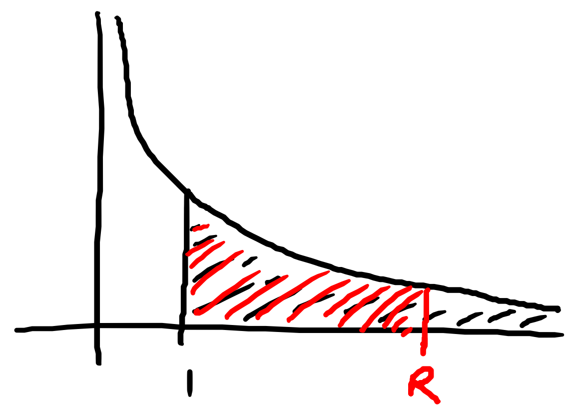
→ If the limit exists, we say that the integral converges. Otherwise, it diverges.

Ex. $\int_1^\infty \frac{1}{x}\,dx =$

$$= \lim_{R\to\infty}\left[\int_1^R \frac{1}{x}\,dx\right] = \lim_{R\to\infty} \ln|x|\Big|_1^R$$

$$= \lim_{R\to\infty}\left[\underbrace{\ln R}_{\infty} - \ln 1\right] = \infty$$

diverges



Ex. $\int_0^{\infty} e^{-x}\,dx$

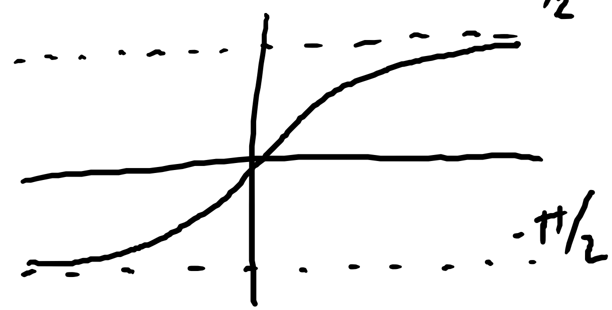
$$= \lim_{R\to\infty} \int_0^R e^{-x}\,dx = \lim_{R\to\infty} -e^{-x}\Big|_0^R$$

$$= \lim_{R\to\infty} \left(-e^{-R} + e^{0}\right) = \lim_{R\to\infty} \left(-\frac{1}{\underbrace{e^R}_{0}} + 1 \right) = 1$$

Ex. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{1+x^2}\,dx$

$$= \lim_{R\to\infty} \tan^{-1}x \Big|_{-R}^{R} = \lim_{R\to\infty} \tan^{-1}R - \tan^{-1}(-R)$$

$$= \left(\frac{\pi}{2}\right) - \left(\frac{-\pi}{2}\right) = \boxed{\pi}$$



$$\underline{\text{Ex.}} \int_1^{\infty} (1-x)e^{-x}\,dx = \lim_{R\to\infty} \int_1^{R} (1-x)e^{-x}\,dx$$

$$= \lim_{R\to\infty} xe^{-x}\Big|_1^R = \lim_{R\to\infty} \left(\underbrace{Re^{-R}}_{0} - 1e^{-1}\right) = \boxed{\frac{-1}{e}}$$

$$\lim_{R\to\infty} Re^{-R} = \lim_{R\to\infty} \frac{R}{e^R} \overset{L}{=} \lim_{R\to\infty} \frac{1}{e^R} = 0$$

$$\lim_{R\to\infty} R = \infty$$

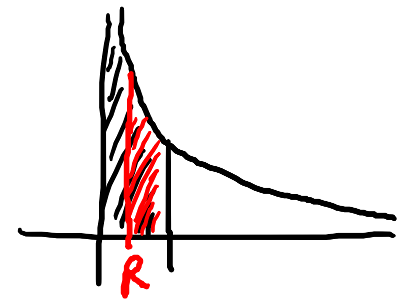
$$\lim_{R\to\infty} e^R = \infty$$

$$\int (1-x)e^{-x}\,dx = (1-x)(-e^{-x}) + \int +e^{-x}(-1)\,dx = -e^{-x} + xe^{-x} + e^{-x} = xe^{-x}$$

$$\boxed{\begin{matrix} u = 1-x & dv = e^{-x}\,dx \\ du = -dx & v = -e^{-x} \end{matrix}}$$

Ex. Find the value of $\int_0^1 \frac{1}{\sqrt[3]{x}}\,dx$ or show that it doesn't exist.

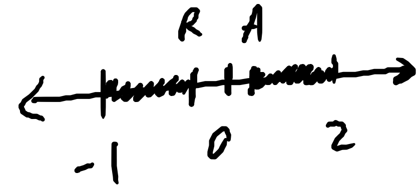
$$= \int_0^1 x^{-1/3}\,dx = \lim_{R\to 0^+} \int_R^1 x^{-1/3}\,dx = \lim_{R\to 0^+} \frac{3}{2} x^{2/3} \Big|_R^1$$



$$= \lim_{R\to 0^+} \left(\frac{3}{2} - \frac{3}{2} R^{2/3} \right) = \frac{3}{2}$$

Note that the statement of the problem clues you in that it's an improper integral.

$$\underline{\text{Ex.}}\int_{-1}^{2}\frac{1}{x^4}\,dx = \int_{-1}^{0}\frac{1}{x^4}\,dx + \int_{0}^{2}\frac{1}{x^4}\,dx$$



$$= \lim_{R\to 0^-}\int_{-1}^{R} x^{-4}\,dx + \lim_{A\to 0^+}\int_{A}^{2} x^{-4}\,dx$$

$$= \lim_{R\to 0^-}\frac{1}{-3}x^{-3}\Big|_{-1}^{R} + \lim_{A\to 0^+} -\frac{1}{3}x^{-3}\Big|_{A}^{2}$$

$$= \lim_{R\to 0^-}\left(\underbrace{\frac{1}{-3R^3}}_{\infty} + \frac{1}{3(-1)^3}\right) + \lim_{A\to 0^+}\left(\frac{-1}{3(2)^3} + \frac{1}{3A^3}\right)$$

Diverge

Ex. $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}(x+1)}\,dx = \int_0^{756} \frac{1}{\sqrt{x}(x+1)}\,dx + \int_{756}^{\infty} \frac{1}{\sqrt{x}(x+1)}\,dx$

$$= \lim_{A\to 0^+} \int_A^{756} \frac{1}{\sqrt{x}(x+1)}\,dx + \lim_{R\to\infty} \int_{756}^{R} \frac{1}{\sqrt{x}(x+1)}\,dx$$

$$= \lim_{A\to 0^+} 2\tan^{-1}\sqrt{x}\,\Big|_A^{756} + \lim_{R\to\infty} 2\tan^{-1}\sqrt{x}\,\Big|_{756}^{R}$$

$$= \lim_{A\to 0^+}\left(\underline{\underline{2\tan^{-1}\sqrt{756}}} - 2\tan^{-1}\sqrt{A}\right) + \lim_{R\to\infty}\left(2\tan^{-1}\sqrt{R} - \underline{\underline{2\tan^{-1}\sqrt{756}}}\right)$$

0

$$= 2\left(\frac{\pi}{2}\right) = \boxed{\pi}$$

$$\int \frac{1}{\sqrt{x}\,(x+1)}\,dx$$

$$\boxed{\begin{aligned} x &= u^2 \\ dx &= 2u\,du \end{aligned}}$$

$u = \sqrt{x}$

$$= \int \frac{2u}{u(u^2+1)}\,du$$

$$= \int \frac{2}{u^2+1}\,du$$

$$= 2\tan^{-1} u$$

$$= 2\tan^{-1}\sqrt{x}$$

$x+1$

$\tan^2\theta + 1$

$x = \tan^2\theta$

$\sqrt{x} = \tan\theta$

Pract.

1. $\displaystyle\int_{1}^{\infty} \frac{1}{x^2}\,dx$ 1

2. $\displaystyle\int_{-\infty}^{0} x e^x\,dx$ –1

3. $\displaystyle\int_{0}^{3} \frac{1}{x-1}\,dx$ diverges

Unit 6 Progress Check: MCQ Part C

- Do them all